This Talk is About ...

- The new description logic $\mathcal{ALC}_{\mathcal{RA}}$
- How to decide (?)
  the concept satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}}$
  - Currently it seems as if $\mathcal{ALC}_{\mathcal{RA}}$ is undecidable !
  - Work in progress
    * Open questions, missing proofs
- Joint-work with Volker Haarslev & Ralf Möller
- Thanks to Anni-Yasmin Turhan, Carsten Lutz,
  & the anonymous reviewers

Syntax of $\mathcal{ALC}_{\mathcal{RA}}$

- Concepts like in $\mathcal{ALC}$
  - $\neg C$, $C_1 \sqcap C_2$, $C_1 \sqcup C_2$, $\exists R.C$, $\forall R.C$
- Satisfiability w.r.t. a set of role axioms = role box $\mathfrak{R}$
  - $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$
  - These are not role value maps!
    - ∗ No composition allowed on the right hand side ("special global" RVMs)
- $\mathfrak{R}$ must be admissible
  - For each R,S at most one role axiom with $R \circ S \sqsubseteq \ldots \in \mathfrak{R}$

Semantics of $\mathcal{ALC}_{\mathcal{RA}}$, Satisfiability

- $C^{\mathcal{I}}$, $R^{\mathcal{I}}$ as usual (see $\mathcal{ALC}$)
- $\mathcal{I} \models C$ iff $C^{\mathcal{I}} \neq \emptyset$
- All roles must be interpreted as disjoint
  - $R, S \in \mathcal{N_R}$, $R \neq S$: $R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$
- $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ iff $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \ldots \cup R_n^{\mathcal{I}}$
- $\mathcal{I} \models \mathfrak{R}$ iff $\forall ra \in \mathfrak{R} : \mathcal{I} \models ra$
- $\mathcal{I} \models (C, \mathfrak{R}_{\mathfrak{C}})$ iff $\mathcal{I} \models C$, $\mathcal{I} \models \mathfrak{R}$

Simple Example

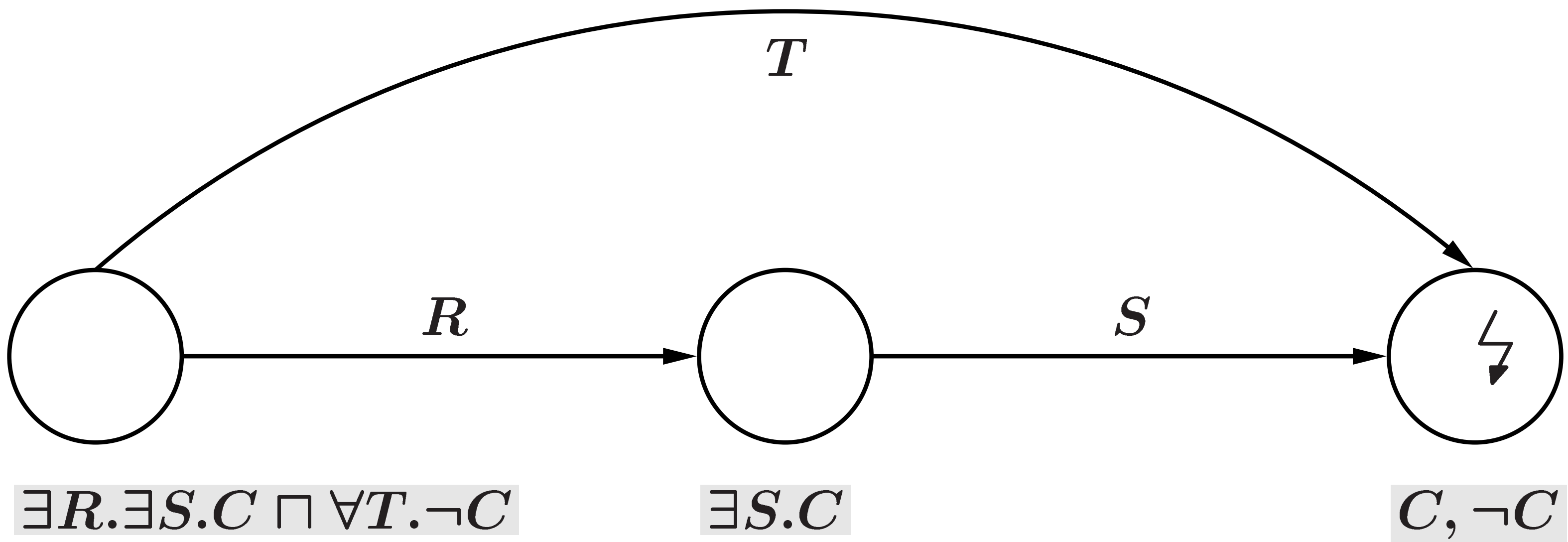
$$((\exists R.\exists S.C) \sqcap \forall T.\neg C, \{R \circ S \sqsubseteq T\})$$

$(\forall[x, y, z](R(x, y) \wedge S(y, z) \Rightarrow T(x, z)))$ **(Role Box)** $\wedge$

$(\forall[x, y](R(x, y) \oplus S(x, y) \oplus T(x, y)))$ **(Disjointness)** $\wedge$

$(\exists[x](\ (\exists[y](R(x, y) \wedge \exists[x](S(y, x) \wedge C(x)))) \wedge$

$(\forall[y](T(x, y) \Rightarrow \neg C(y))))))$ **($\mathcal{ALC}$ Concept, $\in$ monadic $\mathcal{GF}^2$)**

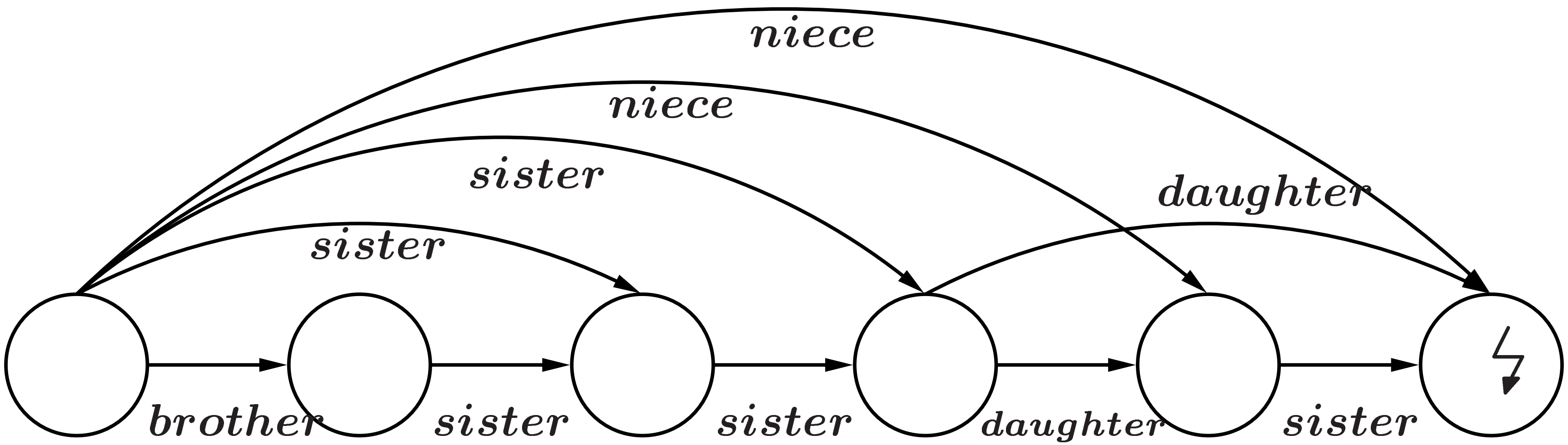


Complex Example

$(\exists brother.\exists sister.\exists sister.\exists daughter.\exists sister.css) \sqcap$
$\quad \forall niece.\neg css$ (**c**omputer **s**cience **s**tudent)

$\{brother \circ sister \sqsubseteq sister, \quad sister \circ daughter \sqsubseteq niece,$
$daughter \circ sister \sqsubseteq daughter, \quad sister \circ sister \sqsubseteq sister\}$

$\lightning \quad niece \sqsubseteq sister \circ daughter \sqcup brother \circ daughter \quad \lightning$

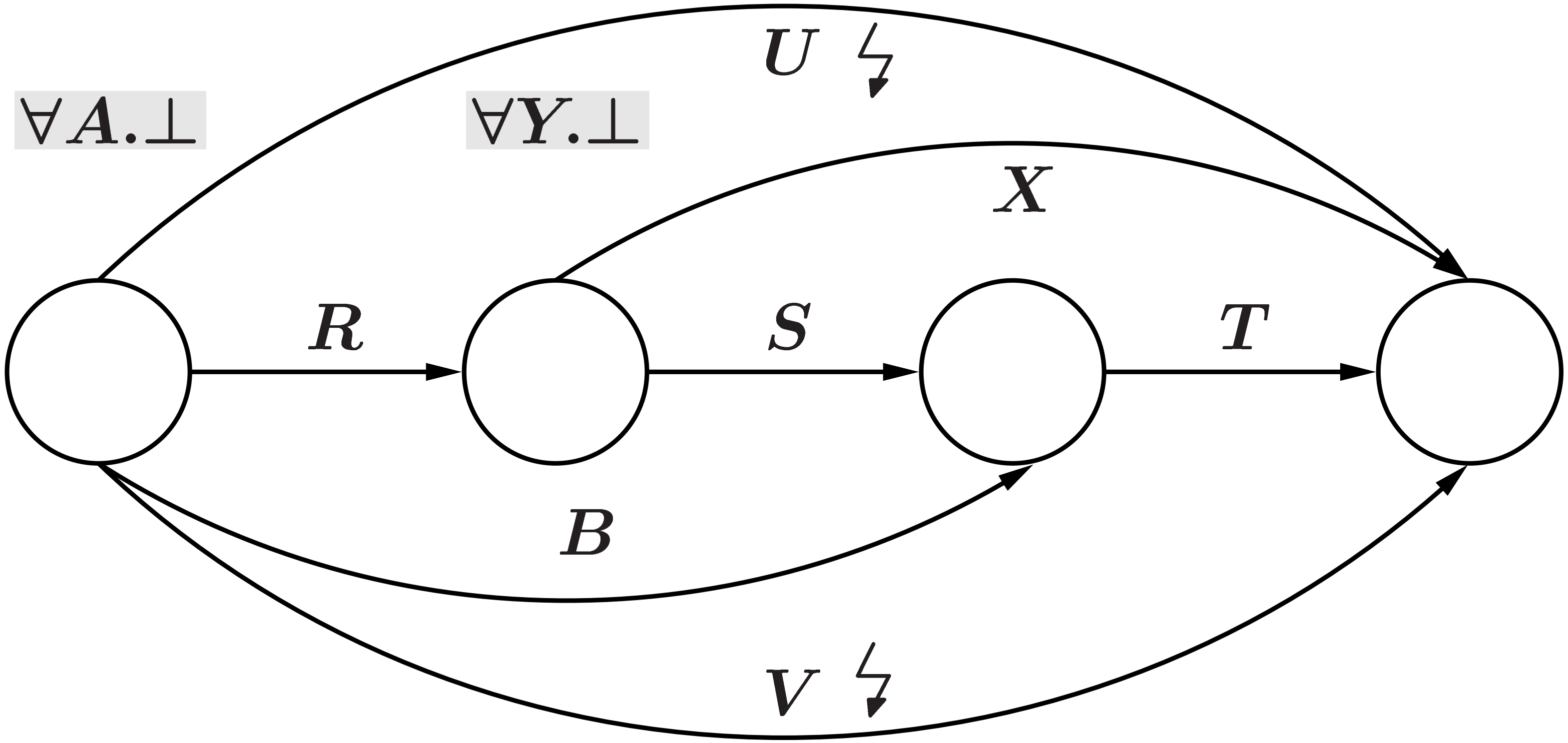


Role Box Clashes

$\exists \boldsymbol{R}.((\exists \boldsymbol{S}.\exists \boldsymbol{T}.\top) \sqcap \forall \boldsymbol{Y}.\bot) \sqcap \forall \boldsymbol{A}.\bot$

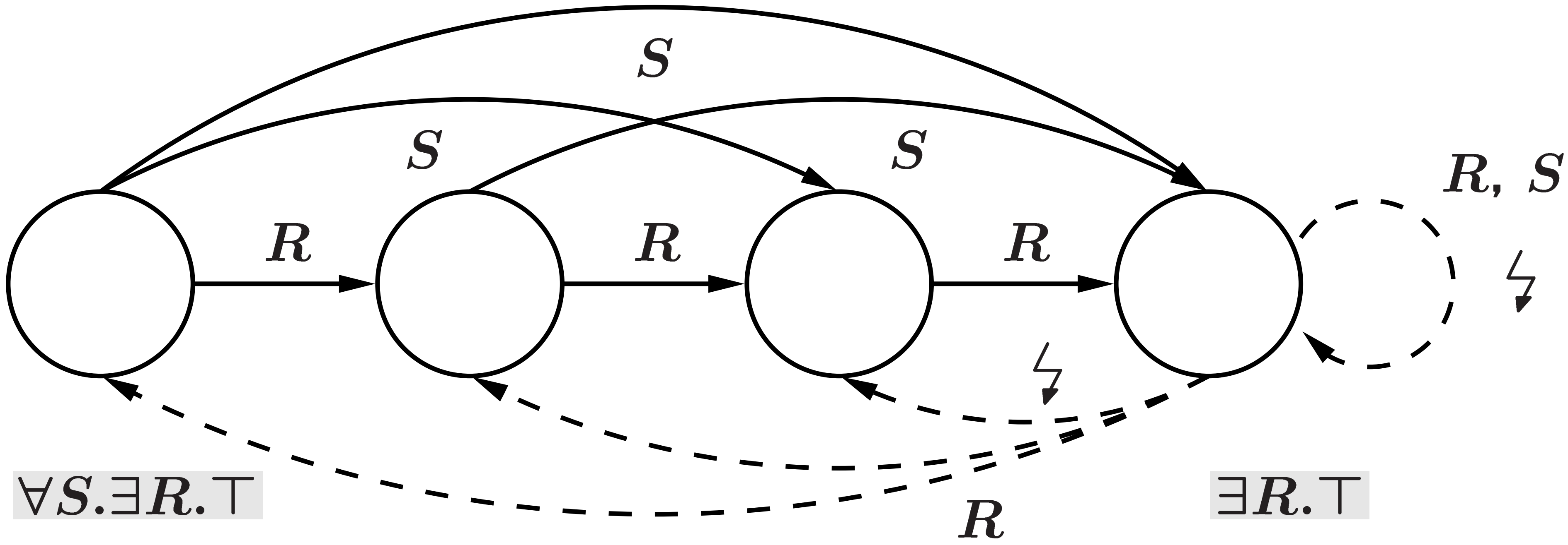
$\{\boldsymbol{R} \circ \boldsymbol{S} \sqsubseteq \boldsymbol{A} \sqcup \boldsymbol{B},\ \ \boldsymbol{S} \circ \boldsymbol{T} \sqsubseteq \boldsymbol{X} \sqcup \boldsymbol{Y},$

$\boldsymbol{A} \circ \boldsymbol{T} \sqsubseteq \boldsymbol{U},\ \ \boldsymbol{B} \circ \boldsymbol{T} \sqsubseteq \boldsymbol{V},\ \ \boldsymbol{R} \circ \boldsymbol{X} \sqsubseteq \boldsymbol{U},\ \ \boldsymbol{R} \circ \boldsymbol{Y} \sqsubseteq \boldsymbol{V}\}$



No Finite Model Property

- Disjoint roles matter (unlike $\mathcal{ALC}$)
- $(\exists R.\exists R.\top) \sqcap (\forall S.\exists R.\top)$ w.r.t. $\{R \circ R \sqsubseteq S,\ \ R \circ S \sqsubseteq S,\ \ S \circ R \sqsubseteq S,\ \ S \circ S \sqsubseteq S\}$



Relationships to Other DLs

- At least as expressive as $\mathcal{ALC}_{\mathcal{R}^+}$ (Sattler)
  - Transitively closed roles, $R \circ R \sqsubseteq R \Rightarrow R^{\mathcal{I}} = (R^{\mathcal{I}})^+$
- At least as expressive as $\mathcal{ALC}_{\oplus}$ (Sattler)
  - "Transitive orbit" operator $\oplus$: $(R^{\mathcal{I}})^+ \subseteq (\oplus(R))^{\mathcal{I}}$
  - $\oplus(R) \rightarrow R_{\oplus}$, $\{R \circ R \sqsubseteq R_{\oplus}, \quad R_{\oplus} \circ R \sqsubseteq R_{\oplus}\} \Rightarrow$ $(\oplus(R))^{\mathcal{I}} = R^{\mathcal{I}} \cup R_{\oplus}^{\mathcal{I}}$

$$\begin{aligned} \exists \oplus (R).C &\rightarrow \exists R_{\oplus}.C \\ \exists R.C &\rightarrow \exists R_{\oplus}.C \sqcap \exists R.C \\ \forall \oplus (R).C &\rightarrow \forall R_{\oplus}.C \sqcap \forall R.C \end{aligned}$$

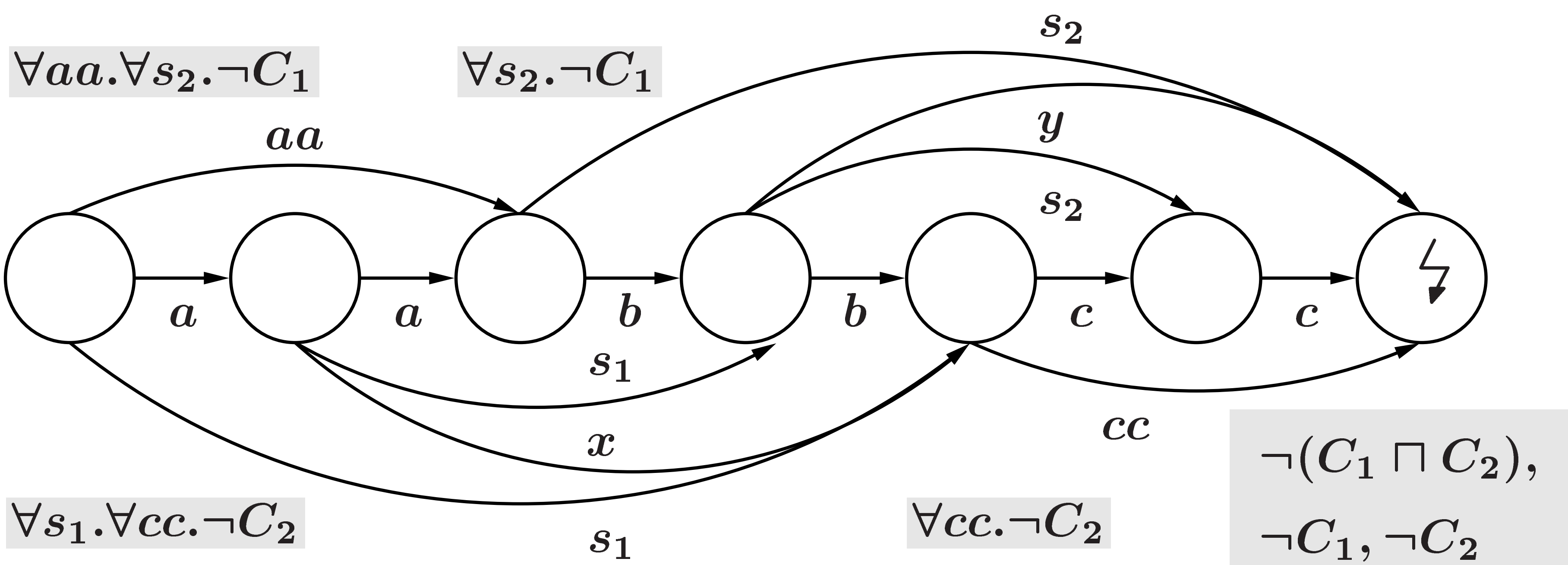
$\Rightarrow$ **EXPTIME**-hardness of $\mathcal{ALC}_{\mathcal{RA}}$

Relationships to Other DLs (2)

- $\mathcal{ALC}_+$ (Baader)
  - Transitive closure operator $+$: $(R^{\mathcal{I}})^+ = (+(R))^{\mathcal{I}}$
  - $\mathcal{ALC}_{\mathcal{RA}} \in \mathcal{FOPL}^3$, $\mathcal{FOPL}^3 \subseteq \mathcal{FOPL}$, but $\mathcal{ALC}_+ \notin \mathcal{FOPL}$
  - $\Rightarrow$ Transitive closure cannot be expressed in $\mathcal{ALC}_{\mathcal{RA}}$
- $\mathcal{ALCH}_{\mathcal{R}^+}$ (Horrocks)
  - Allow non-disjoint roles
  - Allow role inclusion axioms $R \sqsubseteq S \in \mathfrak{R}$
  - $\Rightarrow$ $\mathcal{ALCH}_{\mathcal{R}^+} \subseteq \mathcal{ALCH}_{\mathcal{RA}^\ominus}$

"Accepting" $a^n b^n c^n$

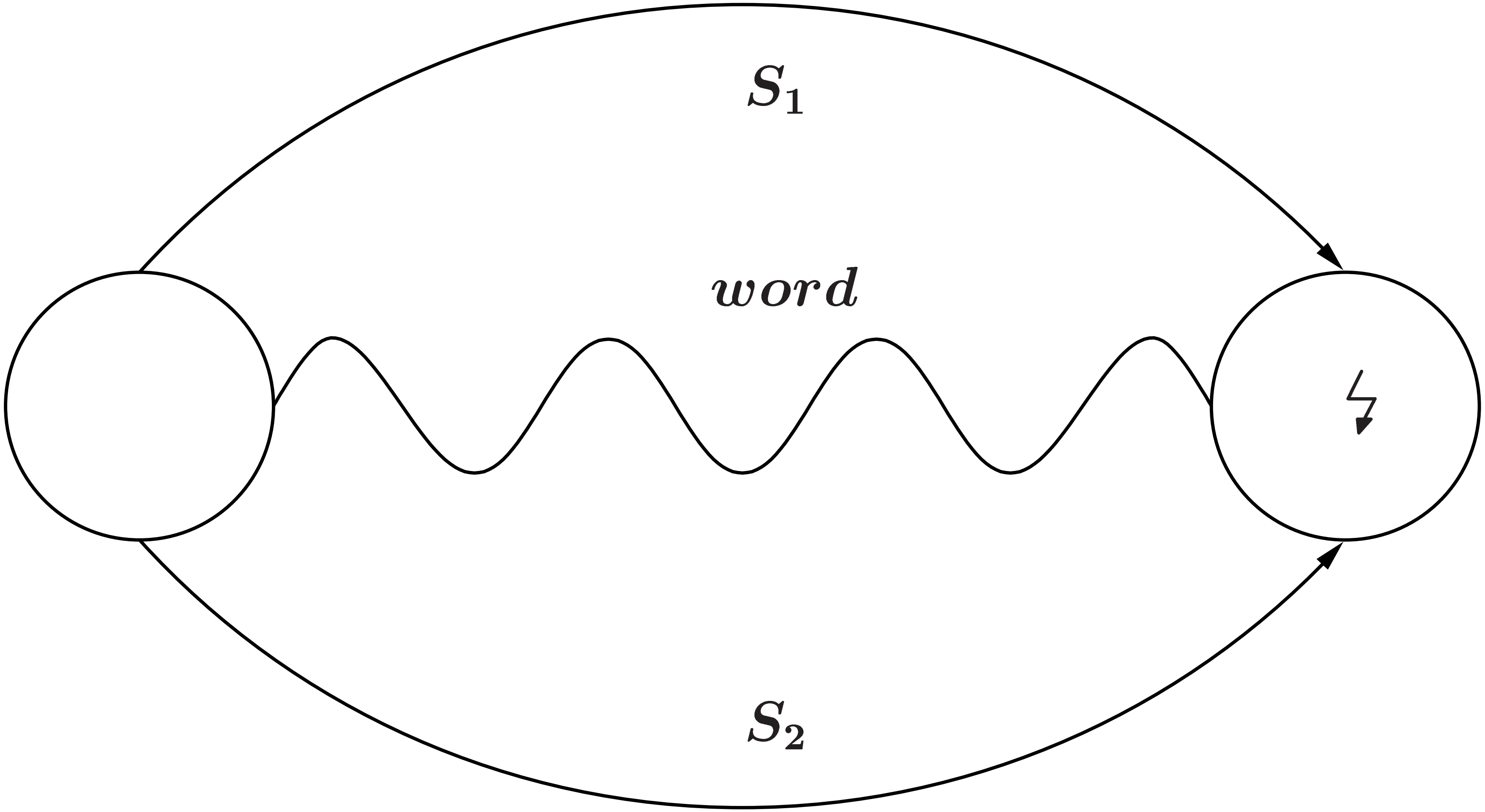
- **SAT**$(\exists word.\neg(C_1 \sqcap C_2) \sqcap \forall s1.((\forall c.\neg C_1) \sqcap (\forall cc.\neg C_1)) \sqcap \forall a.\forall s_2.\neg C_2 \sqcap \forall aa.\forall s_2.\neg C_2)$ **iff** $word \notin \mathcal{L}_{a^n b^n c^n}$
- $\{a \circ x \sqsubseteq s_1,\ a \circ b \sqsubseteq s_1,\ s_1 \circ b \sqsubseteq x,\ b \circ y \sqsubseteq s_2,\ b \circ c \sqsubseteq s_2,\ s_2 \circ c \sqsubseteq y,\ a \circ a \sqsubseteq aa,\ aa \circ a \sqsubseteq aa,\ c \circ c \sqsubseteq cc,\ cc \circ c \sqsubseteq cc\}$



Is $\mathcal{ALC}_{\mathcal{RA}}$ (With Universal Role And Non-Disjoint Roles) Undecidable?

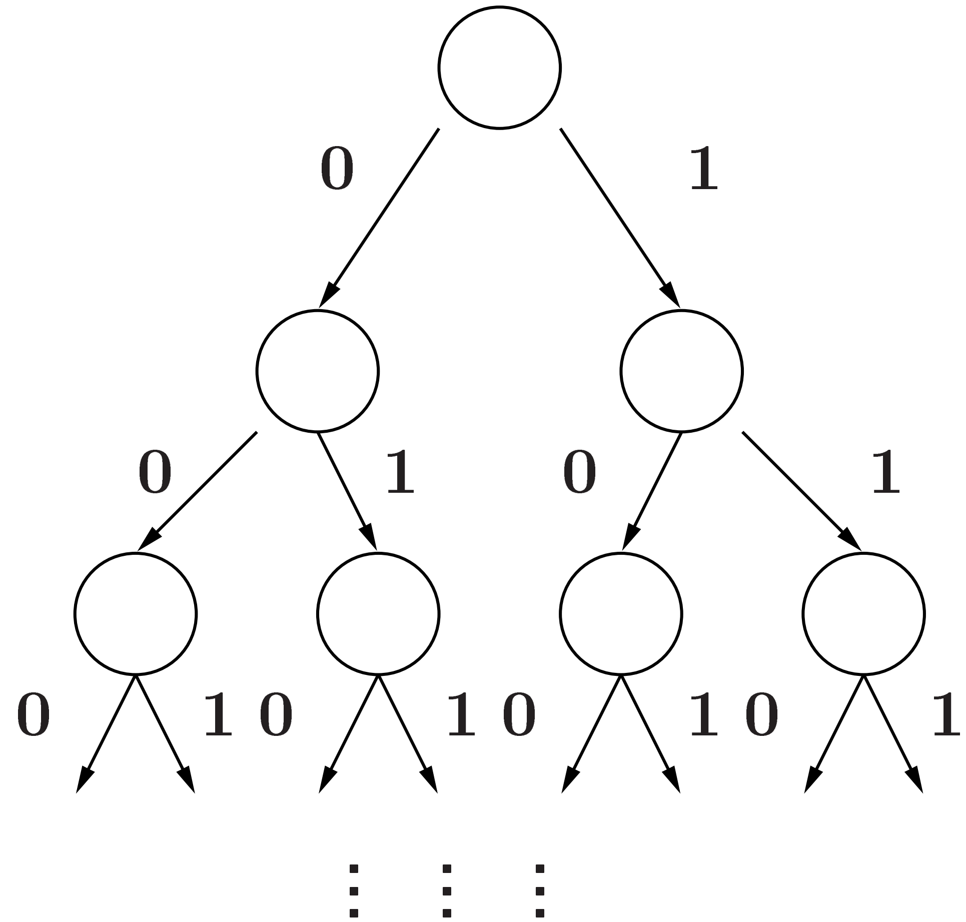
- Transform PCP with $\mathcal{A} = \{0, 1\}$ into two context-free grammars G_1, G_2 with start-symbols S_1, S_2 such that the PCP has a solution iff $\mathcal{L}(G_1) \cap \mathcal{L}(G_2) \neq \emptyset$
- Transform G_1, G_2 into Chomsky Normal Form: G'_1, G'_2
- Transform G'_1, G'_2 into role box $\mathfrak{R}_{G'_1, G'_2}$
- $(\exists word.\neg(C \sqcap D) \sqcap \forall S_1.C \sqcap \forall S_2.D, \mathfrak{R})$ is unsatisfiable iff $word \in \mathcal{L}(G'_1) \cap \mathcal{L}(G'_2)$

Illustration



How to Consider All Words

- Represent $\{0, 1\}^+ = \mathcal{A}^+$ as binary infinite tree (each path has infinite length)
- Sub-paths starting from the root-node correspond to (finite) words $w \in \{0, 1\}^+$
- "$*$" = universal role



$$(((\exists 0.\top) \sqcap (\exists 1.\top) \sqcap (\forall * .((\exists 0.\top) \sqcap (\exists 1.\top)))), \mathfrak{R})$$

Example Reduction

- **PCP** $= ((1, 101), (10, 00), (011, 11))$
- **Solution** $= 1323 = 101110011 = x_1 x_3 x_2 x_3 = y_1 y_3 y_2 y_3$
- $a_3 a_2 a_3 a_1 101110011 \in \mathcal{L}(G_1) \cap \mathcal{L}(G_2)$
- $G_1 = \{\ S_1 \rightarrow a_1 1 \mid a_2 \boxed{10} \mid a_3 \boxed{011} \mid a_1 S_1 1 \mid a_2 S_1 \boxed{10} \mid a_3 S_1 \boxed{011}\ \}$
- $G_2 = \{\ S_2 \rightarrow a_1 \boxed{101} \mid a_2 \boxed{00} \mid a_3 \boxed{11} \mid a_1 S_2 \boxed{101} \mid a_2 S_2 \boxed{00} \mid a_3 S_2 \boxed{11}\ \}$

Example Reduction (2)

- $G'_1 =$

  $\{\ S_1 \rightarrow a_1 1 \mid a_2 \boxed{10} \mid a_3 \boxed{011} \mid a_1 S_{1,1} \mid a_2 S_{1,10} \mid a_3 S_{1,011},$

  $S_{1,1} \rightarrow S_1 1,\quad S_{1,10} \rightarrow S_1 \boxed{10},\quad S_{1,011} \rightarrow S_1 \boxed{011},$

  $\boxed{10} \rightarrow 10,\quad \boxed{11} \rightarrow 11,\quad \boxed{011} \rightarrow 0\boxed{11}\ \}$

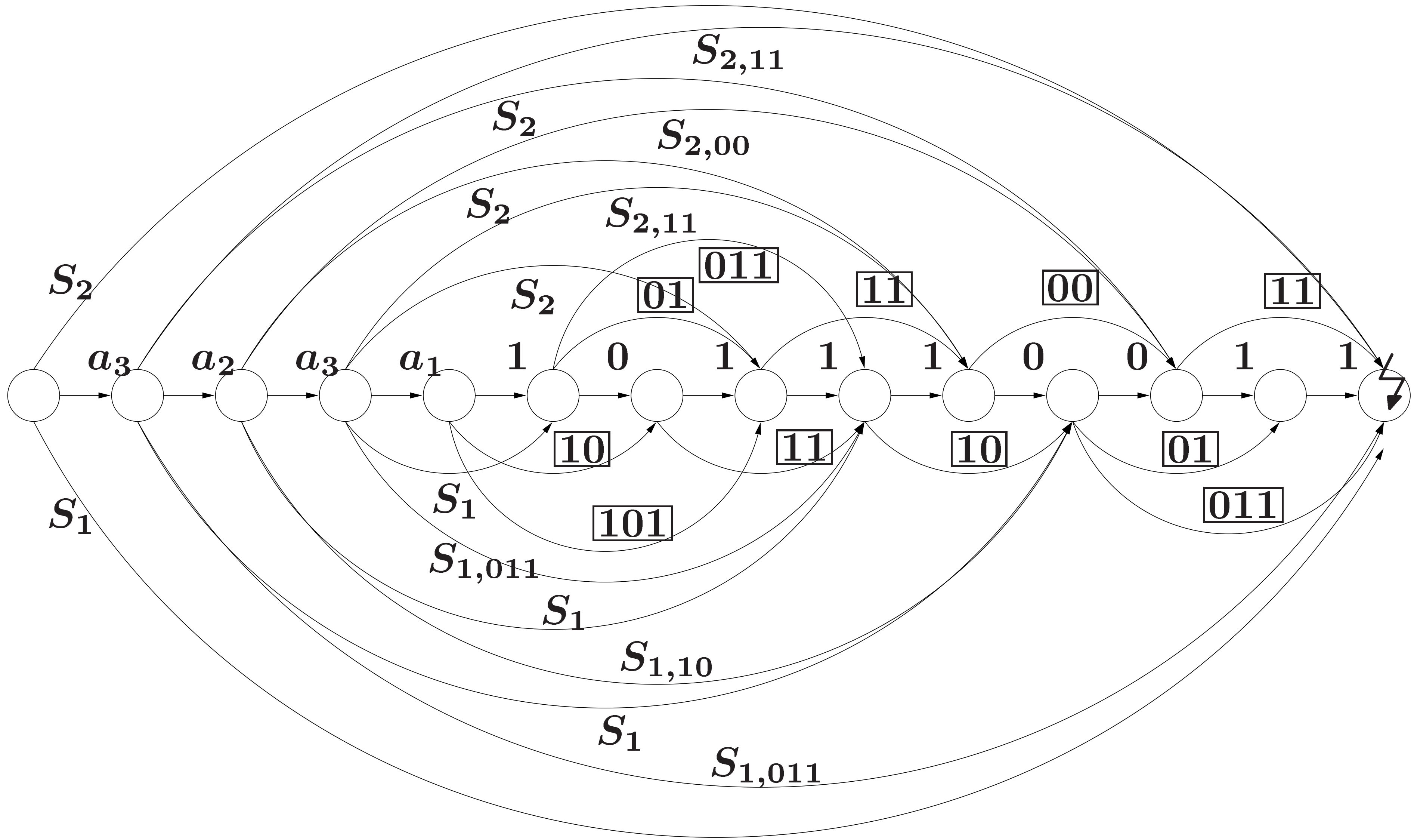
- $G'_2 =$

  $\{\ S_2 \rightarrow a_1 \boxed{101} \mid a_2 \boxed{00} \mid a_3 \boxed{11} \mid a_1 S_{2,101} \mid a_2 S_{2,00} \mid a_3 S_{2,11},$

  $S_{2,101} \rightarrow S_1 \boxed{101},\quad S_{2,00} \rightarrow S_1 \boxed{00},\quad S_{2,11} \rightarrow S_1 \boxed{11},$

  $\boxed{00} \rightarrow 00,\quad \boxed{11} \rightarrow 11,\quad \boxed{01} \rightarrow 01,\quad \boxed{101} \rightarrow 1\boxed{01}\ \}$

- “Reverse” all productions $\Rightarrow \mathfrak{R}_{G'_1, G'_2}$
- All terminal and non-terminal symbols are roles



$S_{2,11}$
S_2
$S_{2,00}$
S_2
$S_{2,11}$
011
S_2
01
11
00
11
S_2
a_3
a_2
a_3
a_1
1
0
1
1
1
0
0
1
1
10
11
10
01
S_1
S_1
101
011
$S_{1,011}$
S_1
$S_{1,10}$
S_1
$S_{1,011}$

Example Reduction (3)

Claim:

$$\exists\{0,1,a_1,a_2,a_3\}.\neg(C \sqcap D) \sqcap$$
$$\forall * .(\exists\{0,1,a_1,a_2,a_3\}.\neg(C \sqcap D)) \sqcap$$
$$\forall S_1.C \sqcap \forall S_2.D$$

w.r.t.

$\mathfrak{R}_{\mathcal{C}(G'_1,G'_2)}$

is satisfiable

iff

the PCP has no solution