Obstacles on the Way to Spatial Reasoning with Description Logics – Some Undecidability Results

- **Motivation & Example**
- **Undecidability of $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$ (DL '2000)**
- **Decidability of $\mathcal{ALC}_{\mathcal{RASG}}$**
- **Undecidability of $\mathcal{ALCF}_{\mathcal{RASG}}$**
- **Undecidability of $\mathcal{ALC}_{\mathcal{RA}}$**
- **Future Work**

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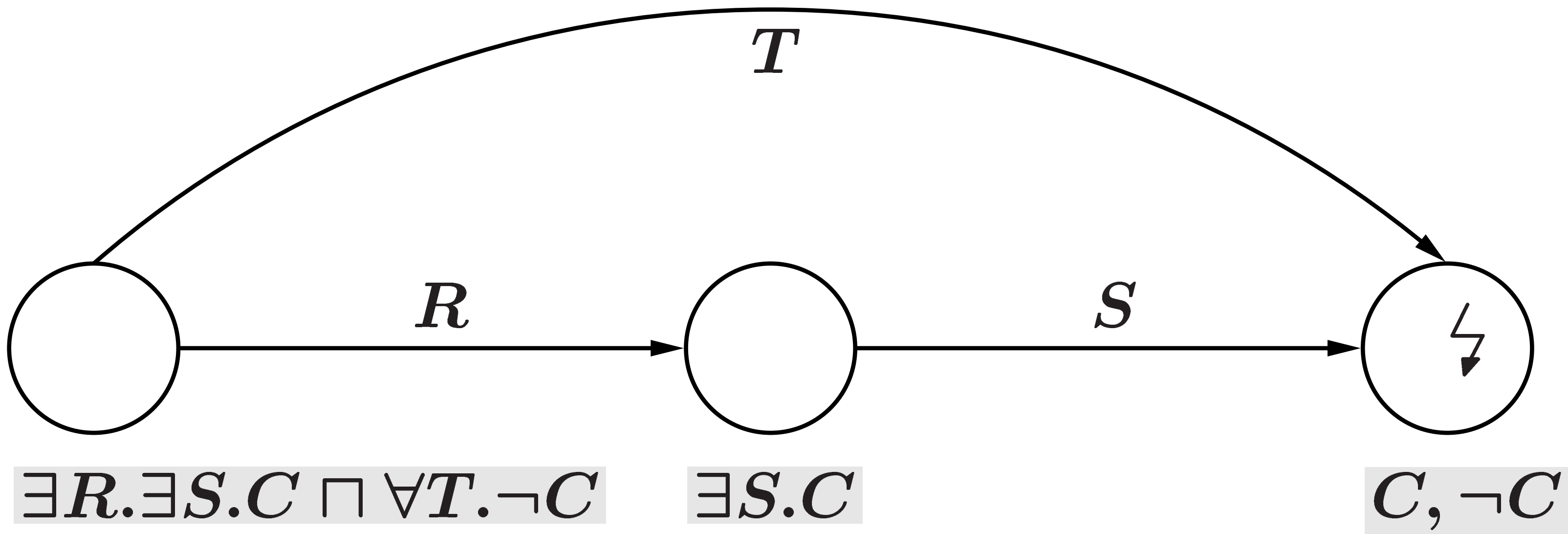
$\mathcal{ALC}_{\mathcal{RA}^\ominus}$ and $\mathcal{ALC}_{\mathcal{RA}}$

- $\mathcal{ALC}$ extended with generalized transitivity axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$
- Semantics:
  $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ iff
  $$S^\mathcal{I} \circ T^\mathcal{I} \subseteq R_1^\mathcal{I} \cup \cdots \cup R_n^\mathcal{I}$$
  ("$\circ$" $=$ relational composition)
- Set of these axioms: role box $\mathfrak{R}$
- "Qualitative Composition-Table Based Reasoning"
- $\mathcal{ALC}_{\mathcal{RA}}$: for all R,S, $R \neq S$ enforce $R^\mathcal{I} \cap S^\mathcal{I} = \emptyset$

$\Rightarrow$ "Disjoint Spatial Base Relations"

- Cohn '92 ($\Box_{EC}, \Diamond_{EC}, \ldots$), HLM '98 ($\mathcal{ALCRP}(\mathcal{S}_2)$)

$((\exists R.\exists S.C) \sqcap \forall T.\neg C, \{R \circ S \sqsubseteq T\})$



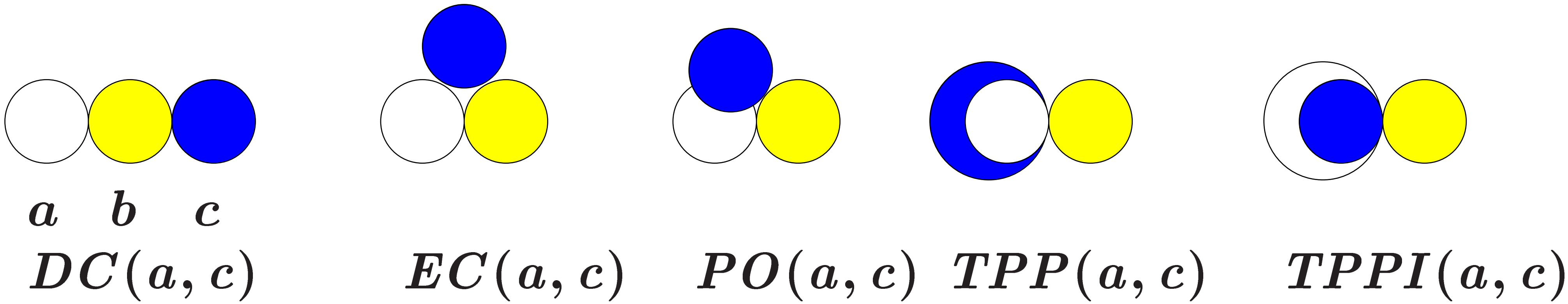


Illustration of the composition table entry EC $\times$ EC

$$\forall x, y, z : EC(x, y) \wedge EC(y, z) \Rightarrow (DC(x, z) \vee EC(x, z) \vee PO(x, z) \vee TPP(x, z) \vee TPPI(x, z))$$

$$EC \circ EC \sqsubseteq DC \sqcup EC \sqcup PO \sqcup TPP \sqcup TPPI$$

$$
\begin{array}{lcl}
circle & \dot{\sqsubseteq} & figure \\
figure\_touching\_a\_figure & \doteq & figure \sqcap \exists EC.figure \\
special\_figure & \doteq & figure \sqcap \\
 & & \forall PO.\neg figure \sqcap \\
 & & \forall NTPPI.\neg figure \sqcap \\
 & & \forall TPPI.\neg circle \sqcap \\
 & & \exists TPPI.(figure \sqcap \exists EC.circle)
\end{array}
$$

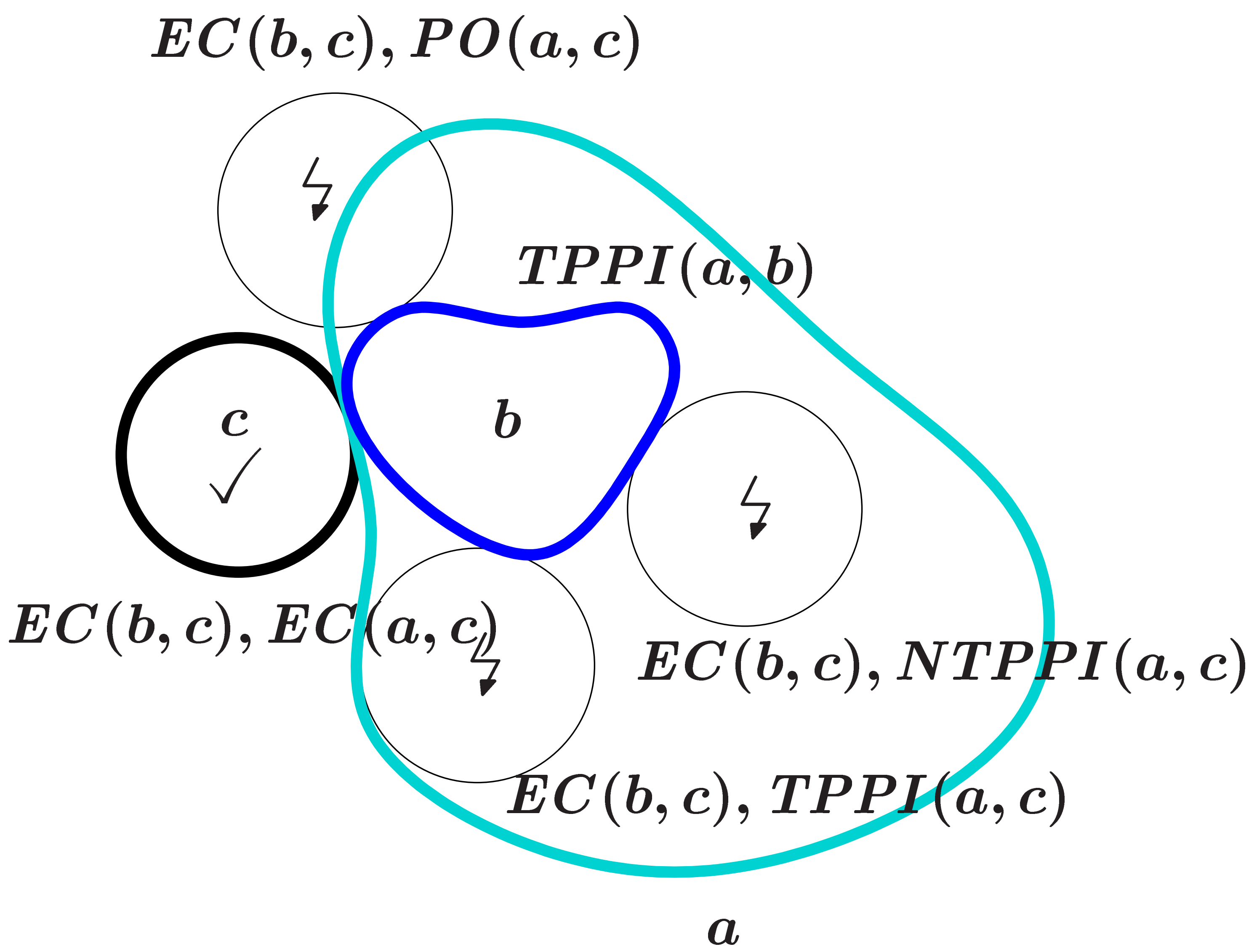
$special\_figure \sqsubseteq figure\_touching\_a\_figure$ **iff**

$figure \sqcap \forall PO.\neg figure \sqcap \forall NTPPI.\neg figure \sqcap \forall TPPI.\neg circle \sqcap \exists TPPI.(figure \sqcap \exists EC.circle) \sqcap \neg(figure \sqcap \exists EC.figure)$

is unsatisfiable w.r.t.

$\mathfrak{R} = \{\ldots, TPPI \circ EC \sqsubseteq EC \sqcup PO \sqcup TPPI \sqcup NTPPI, \ldots\}$

Illustration of $\mathcal{I} \models special\_figure$



How to Enforce Non-Empty Role Intersections Slide 7

$(\exists R.\exists S.\top, \quad \{R \circ S \sqsubseteq T, R \circ S \sqsubseteq U\})$

- **Satisfiable in $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$**
- **Unsatisfiable in $\mathcal{ALC}_{\mathcal{RA}}$ due to $T^{\mathcal{I}} \cap U^{\mathcal{I}} \neq \emptyset$**

$\Rightarrow$ **Allow only "functional" role boxes**

$(\ \exists R.\exists S.\exists T.\top,$
$\{R \circ S \sqsubseteq RS, S \circ T \sqsubseteq ST, R \circ ST \sqsubseteq U, RS \circ T \sqsubseteq V\})$

- **Satisfiable in $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$**
- **Unsatisfiable in $\mathcal{ALC}_{\mathcal{RA}}$ due to $U^{\mathcal{I}} \cap V^{\mathcal{I}} \neq \emptyset$**
- $(R \circ S) \circ T = U \neq V = R \circ (S \circ T)$

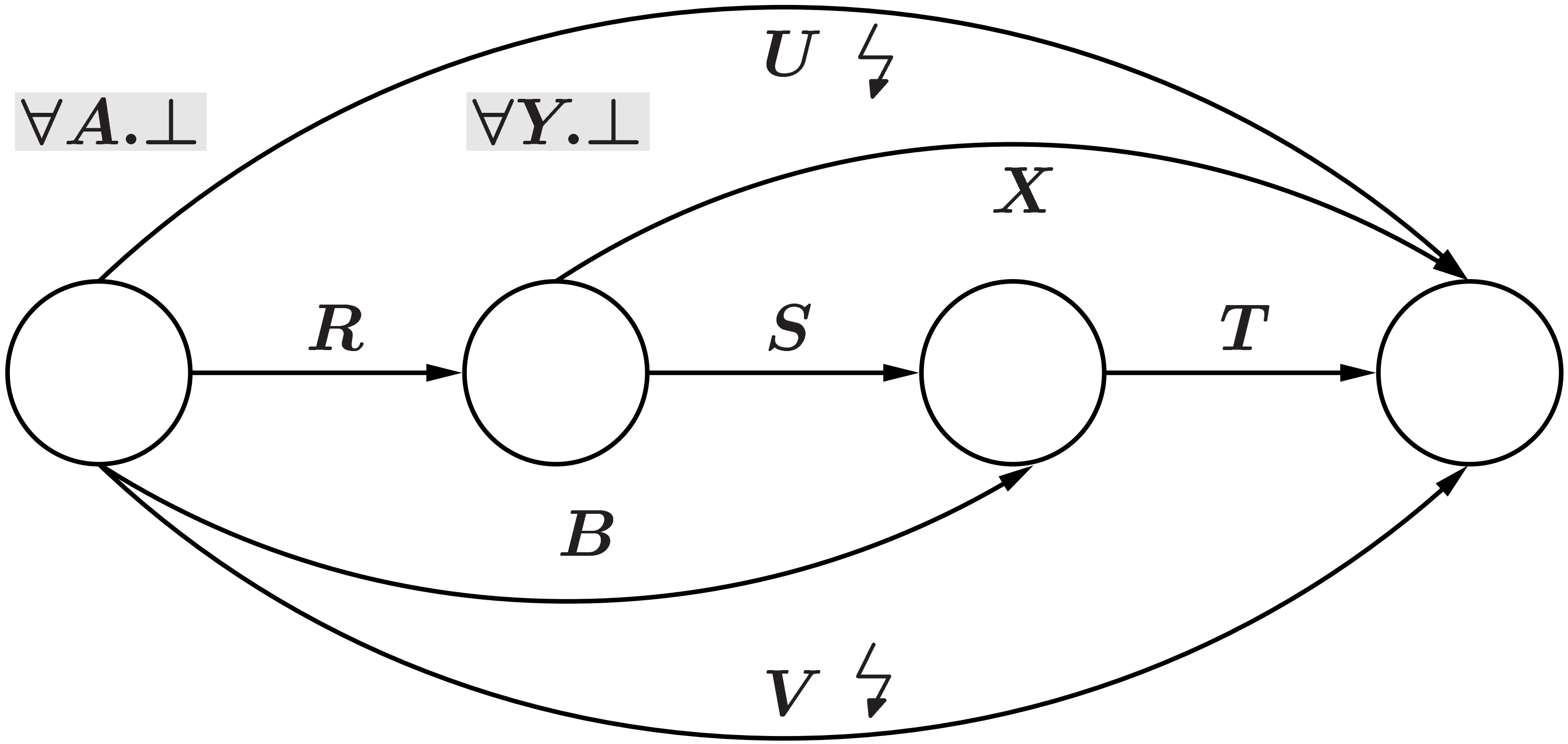
$\Rightarrow$ **Allow only "associative" role boxes**

In this example, $(R \circ S) \circ T = \{U, V\} = R \circ (S \circ T)$

$\exists R.((\exists S.\exists T.\top) \sqcap \forall Y.\bot) \sqcap \forall A.\bot$

$\{R \circ S \sqsubseteq A \sqcup B, \quad S \circ T \sqsubseteq X \sqcup Y,$

$A \circ T \sqsubseteq U, \quad B \circ T \sqsubseteq V, \quad R \circ X \sqsubseteq U, \quad R \circ Y \sqsubseteq V\}$



- In the previous example we had
  $U^{\mathcal{I}} \cap V^{\mathcal{I}} \neq \emptyset$ due to
  $\text{con}(R, X) = U$ and $\text{con}(B, T) = V$
  $\Rightarrow$ $\text{con}(R, X) \cap \text{con}(B, T) = \emptyset$
- Let $w \in \mathcal{N_R}^\star$ (in the Example: $w = RST$), let $\mathcal{COMP}_i$ be a complete set of role tuples that can be build "on" w (in the Example:
  $\mathcal{COMP}_1 = \{(A, T), (R, X)\}$,
  $\mathcal{COMP}_2 = \{(B, T), (R, X)\}, \ldots$):
  - If for some $\mathcal{COMP}$, $\bigcap_{(R,S) \in \mathcal{COMP}} (\text{con}(R, S)) = \emptyset$,

$\Rightarrow$ Non-empty role intersections can be enforced

- $\mathcal{ALC}_{\mathcal{RA}\ominus}$ is undecidable
  - Proof sketch at DL 2000
  - "Context-Free Inclusion Modal Logics" (Baldoni '98)
  - "Grammar Logics"
- Special "admissible classes" of role boxes are decidable, e.g. $\mathcal{ALC}$, $\mathcal{ALC}_{\mathcal{R}+}$, $\mathcal{ALC}_{\mathcal{R}\oplus}$
  - Logics with some kind of "Tree Model Property"
  - Disjoint roles don't matter w.r.t. satisfiability
  - Role Disjointness is not modally definable (e.g. in $\mathcal{ALC}$)
- $\mathcal{ALC}_{\mathcal{RASG}}$ is also decidable (see below)

Undecidability of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$

- **Reduction from the intersection problem of context-free grammars**
- **Each "reversed production" in a NF similar to CNF gives rise to one role axiom:** $A \rightarrow BC$ $\rightsquigarrow$ $B \circ C \sqsubseteq A$
- $(E, \mathfrak{R}')$ **is satisfiable iff** $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$

$$E =_{def} X \sqcap \neg(C \sqcap D) \sqcap Y \sqcap \forall S_1.C \sqcap \forall S_2.D$$

$$X =_{def} \sqcap_{a \in \Sigma} \exists a.\top$$

$$Y =_{def} \sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.(X \sqcap \neg(C \sqcap D))$$

$$\mathfrak{R}' =_{def} \mathfrak{R} \cup \{R \circ S \sqsubseteq R_* \mid R, S \in (\{R_*\} \cup \mathsf{roles}(\mathfrak{R})), \neg\exists ra \in \mathfrak{R} : \mathsf{pre}(ra) = (R, S)\}$$

Questions Concerning $\mathcal{ALC}_{\mathcal{RCC}\,8}$

- **Are $\mathcal{ALC}_{\mathcal{RCC}\,8}$ and $\mathcal{ALC}_{\mathcal{RCC}\,5}$ decidable?**
- **Do disjoint roles matter w.r.t. satisfiability?**

  $\Rightarrow \mathcal{ALC}_{\mathcal{RCC}\,8} = \mathcal{ALC}_{\mathcal{RCC}\,8^{\ominus}}$?

  - **at least no counter-example is known!**
  - **Claim: disjoint roles make no difference; non-empty role intersections cannot be enforced by concept terms (unlike example on Slide 8)**
- **What about inverse roles ($\mathcal{ALCI}_{\mathcal{RCC}\,8}$)?**
- **Structure in the composition tables**
  - **Associativity, Symmetry, Helly-Property, ...**
  - **Calculus of binary relations is undecidable ($\mathcal{FOPL}_3$)**

“Solutions” to Overcome the Undecidability

- $|\Delta^{\mathcal{I}}| < magical\_number$
- **Remove “all” quantification**
  - **Design a logic in the style of $\mathcal{TL} - \mathcal{ALCF}$, use variables etc.**
- **Make syntax-restrictions like in $\mathcal{ALCRP}(\mathcal{D})$**
  - **Disallow quantifier patterns “$\forall \ldots \exists$” and “$\exists \ldots \forall$”**
  - **even $\mathcal{ALCI}_{\mathcal{RA}^{\ominus}}$ and $\mathcal{ALCI}_{\mathcal{RA}}$ would probably work**

$\Rightarrow$ **Modeling interesting concepts becomes harder**

$\mathcal{ALC}_{\mathcal{RASG}}$ is Decidable

- Like $\mathcal{ALC}_{\mathcal{RA}\ominus}$, but allow only admissible role boxes
  - No disjunctions in the role axioms in $\mathfrak{R}$
  - for all roles $R, S \in$ roles($\mathfrak{R}$) there is exactly one role axiom with $R \circ S \sqsubseteq \ldots \in \mathfrak{R}$ ("functional role boxes")
  - Associativity: $\forall R, S, T \in$ roles($\mathfrak{R}$) : con(con(R, S), T) = con(R, con(S, T))

  $\Rightarrow$ Each composition possibility yields the same role

- Examples
  - $\{R \circ R \sqsubseteq R\}$ ($\mathcal{ALC}_{\mathcal{R}^+} \subseteq \mathcal{ALC}_{\mathcal{RASG}}$)
  - Expanded RCC8/RCC5 composition table
  - Semi-Groups like ($\mathbb{N}\ mod\ n, +\ mod\ n$)

Reduction from $\mathcal{ALC}_{\mathcal{RASG}}$ to $\mathcal{ALC}$ & GCIs

- Similar like for $\mathcal{ALC}_{\mathcal{R}^+}$;
  for a transitively closed role R add
  - the GCI $\forall R.C \Rightarrow \forall R.\forall R.C$
  - resp. "K4" $\Box_R C \Rightarrow \Box_R \Box_R C$
  - for all "relevant" concepts C
- $\mathcal{ALC}_{\mathcal{RASG}}$: For each relevant "$\forall R.C$" add
  "Initialize:" $\forall R.C \sqsubseteq \boxed{\forall R.C}$ (concept name)
  "Apply": $\boxed{\forall R.C}_R \sqsubseteq C$
  "Propagate": $\boxed{\forall R.C}_S \sqsubseteq \forall T.\boxed{\forall R.C}_{\mathsf{con}(S,T)}$
- Tableaux-calculus for $\mathcal{ALC}_{\mathcal{RASG}}$: propagation of $\boxed{\forall R.C}_S$ concepts, similar like in $\mathcal{ALC}_{\mathcal{R}^+}$ calculus

- **Reduction from the Domino Problem**
- $\mathcal{DOM} =_{def} (\mathcal{D}, \mathcal{H}, \mathcal{V})$
  - $\mathcal{D} = \{d_1, \ldots, d_n\}$ **Domino Types**
  - $\mathcal{V} \subseteq \mathcal{D} \times \mathcal{D}$ **vertical matching relation**
  - $\mathcal{H} \subseteq \mathcal{D} \times \mathcal{D}$ **horizontal matching relation**
- **A solution of $\mathcal{DOM}$ is a total function f $: \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{D}$ ($0 \in \mathbb{N}$) such that for all $(i, j) \in \mathbb{N} \times \mathbb{N}$:**
  - $(\mathsf{f}(i, j), \mathsf{f}(i + 1, j)) \in \mathcal{H}$
  - $(\mathsf{f}(i, j), \mathsf{f}(i, j + 1)) \in \mathcal{V}$

$\circ$	R_X	R_Y	R_Z	R_U
R_X	R_U	R_Z	R_U	R_U
R_Y	R_Z	R_U	R_U	R_U
R_Z	R_U	R_U	R_U	R_U
R_U	R_U	R_U	R_U	R_U

$$C =_{def} X \sqcap (\forall R_X.X) \sqcap (\forall R_Y.X) \sqcap (\forall R_Z.X) \sqcap (\forall R_U.X)$$

$$X =_{def} M \sqcap (\geq R_X\ 1) \sqcap (\geq R_Y\ 1) \sqcap (\leq R_X\ 1) \sqcap (\leq R_Y\ 1) \sqcap (\leq R_Z\ 1)$$

$$M =_{def} \sqcup_{D_i \in \mathcal{D}}(D_i \sqcap (\sqcap_{D_j \in \mathcal{D}, D_i \neq D_j} \neg D_j)) \sqcap \sqcap_{D_i \in \mathcal{D}}(D_i \Rightarrow (\forall R_X.(\sqcup_{(D_i,D_j) \in \mathcal{H}} D_j) \sqcap \forall R_Y.(\sqcup_{(D_i,D_j) \in \mathcal{V}} D_j)))$$

- Undecidability proof of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ does not work:
  - "$C \sqcap D$" for some node x iff $\langle root, x \rangle \in S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}}$
  - $E =_{def} X \sqcap \neg(C \sqcap D) \sqcap Y \sqcap \forall S_1.C \sqcap \forall S_2.D$
- Construct $\mathcal{G}'_1$ and $\mathcal{G}'_2$ from $\mathcal{G}_1$ and $\mathcal{G}_2$ such that
  $a_1 a_2 \ldots a_n \in \mathcal{L}(\mathcal{G}_1)$ iff $\# a_1 \# a_2 \ldots \# a_n \in \mathcal{L}(\mathcal{G}'_1)$
  $a_1 a_2 \ldots a_n \in \mathcal{L}(\mathcal{G}_2)$ iff $a_1 \# a_2 \# \ldots a_n \# \in \mathcal{L}(\mathcal{G}'_2)$

$$\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) \neq \emptyset \text{ iff}$$
$$a_1 a_2 \ldots a_n \in \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) \text{ iff}$$
$$\# a_1 \# a_2 \# \ldots \# a_n \# \in \mathcal{L}(\mathcal{G}'_1) \cap \mathcal{L}(\mathcal{G}'_2) \text{ iff}$$
$$(\{\#\}\mathcal{L}(\mathcal{G}'_1)) \cap (\mathcal{L}(\mathcal{G}'_2)\{\#\}) \neq \emptyset$$

$$(E, \mathfrak{R})$$

is satisfiable

iff

$$(\{\#\}\mathcal{L}(\mathcal{G'}_1)) \cap (\mathcal{L}(\mathcal{G'}_2)\{\#\}) = \emptyset$$

where

$$E =_{def} X \sqcap \neg(C \sqcap D) \sqcap Y \sqcap (\forall \#.\forall S_1.C) \sqcap (\forall S_2.\forall \#.D)$$

$$X =_{def} \sqcap_{a \in \Sigma} \exists a.\top$$

$$Y =_{def} \sqcap_{R \in \mathsf{roles}(\mathfrak{R})} \forall R.(X \sqcap \neg(C \sqcap D))$$

- **Infinite "spatial models" are useless for applications:**
  $circle \sqcap (\exists PP.circle) \sqcap (\forall PP.\exists PP.circle)$
  ("PP" = Proper Part)
  - **"infinite descending chain of circles" $\Rightarrow$ unwanted!**
  - **cyclical models that contain "PP" loops $\Rightarrow$ unwanted!**
  - **wanted: finite models that satisfy $Id(\Delta^{\mathcal{I}})^{\mathcal{I}} \subseteq EQ^{\mathcal{I}}$**

  $\Rightarrow$ **under this interpretation,**

  $$circle \sqcap (\exists PP.circle) \sqcap (\forall PP.\exists PP.circle)$$

  is unsatisfiable

- **Can the concept terms that are only satisfiable in infinite models be recognized?**