On Spatial Reasoning with Description Logics

- Motivation
- The family of $\mathcal{ALCI}_{\mathcal{RCC}}$ logics
- Work in progress
  - What we know
  - What we don’t know
- Future Work
Motivation

- We want a DL for “qualitative composition-table based spatial reasoning” in the style of $\mathcal{ALC}_R P(S_2)$, but without syntax-restrictions (if possible)
- With roles corresponding to RCC relationships
- Cohn ’93: Multi-modal spatial logic with “$\square_R, \Diamond_R$” for each RCC-relationship $R$
- Purely relational semantics
  (no truly spatial interpretations yet)
- Related to Relation Algebras, but weaker semantics
  (e.g., our models must not necessarily be representations of finite relation algebras)
The $\mathcal{ALC I}_{\text{RCC}}$-family

- We are considering this problem in a DL-setting.
- In contrast to previous work: inverse roles.
- $\mathcal{ALC I}$ with disjoint roles and global role axioms of the form $S \circ T \subseteq R_1 \sqcup \cdots \sqcup R_n$.
- Semantics:
  \[ \mathcal{I} \models S \circ T \subseteq R_1 \sqcup \cdots \sqcup R_n \quad \text{iff} \quad S^\mathcal{I} \circ T^\mathcal{I} \subseteq R_1^\mathcal{I} \sqcup \cdots \sqcup R_n^\mathcal{I} \]
- With role boxes corresponding to RCC1, RCC2, RCC3, RCC5, RCC8: "$\mathcal{ALC I}_{\text{RCC}}$-family",
  $\mathcal{ALC I}_{\text{RCC}1}$, $\mathcal{ALC I}_{\text{RCC}2}$, $\ldots$, $\mathcal{ALC I}_{\text{RCC}8}$
- With arbitrary role boxes: undecidable
  (representability of Relation Algebras is undecidable)

Michael Wessel, April 2002
Given $EC(a, b)$, $EC(b, c)$, what do we know about the relationship between $a$ and $c$? Lookup $EC \circ EC$ in the RCC8 composition-table:

\[ \forall x, y, z : EC(x, y) \land EC(y, z) \Rightarrow (DC(x, z) \lor EC(x, z) \lor PO(x, z) \lor TPP(x, z) \lor TPPI(x, z)) \]

$EC \circ EC \subseteq DC \sqcup EC \sqcup PO \sqcup TPP \sqcup TPPI$
\begin{align*}
circle & \subseteq figure \\
figure \text{touching}_a\_figure & \overset{\text{def}}{=} figure \cap \exists EC. figure \\
\text{special}_\text{figure} & \overset{\text{def}}{=} figure \cap \\
& \forall PO. \neg figure \cap \\
& \forall NTPPI. \neg figure \cap \\
& \forall TPPI. \neg circle \cap \\
& \exists TPPI. (figure \cap \exists EC. circle)
\end{align*}

\text{special}_\text{figure} \sqsubseteq \text{figure}_\text{touching}_a\_figure \iff

\begin{align*}
figure \cap \forall PO. \neg figure \cap \forall NTPPI. \neg figure \cap \forall TPPI. \neg circle \cap \\
& \exists TPPI. (figure \cap \exists EC. circle) \cap \neg (figure \cap \exists EC. figure)
\end{align*}

is unsatisfiable w.r.t.

\[ \mathcal{K} = \{ \ldots, TPPI \circ EC \sqsubseteq EC \sqcup PO \sqcup TPPI \sqcup NTPPI, \ldots \} \]
• “RCC1”: Only one spatial role $SR$, “spatially related”

• Composition table: $\{SR \circ SR \rightarrow SR\}$

• $SR$ is an equivalence relation

• Equivalent to modal logic “S5”

• “S5” reduction principles:

  $\Box p \equiv \Box \Box p$, $\Box p \equiv \Diamond \Box p$, $\Diamond p \equiv \Diamond \Diamond p$, $\Box p \equiv \Box \Box p$

  $\Rightarrow$ nested occurrences of modalities can be flattened

• NP-complete satisfiability problem
\[ \exists O.C \Rightarrow \forall O. (C \cup \exists \{O, DR\}.C) \cap \forall DR. \exists \{O, DR\}.C \]

- Instead of reduction principles, we have axioms like

- Complexity?

- “RCC2”: reflexive, symmetric role \( O = \text{“overlap”} \), irreflexive and symmetric role \( DR = \text{“discrete from”} \)

- Models are fairly trivial: each complete random graph with \( \text{Id}(\Delta^x) \subseteq O^x \) is a model of the role box
$\mathcal{ALCI}_{RCC3} \ldots \mathcal{ALCI}_{RCC8}$: Role Constraints

- $\geq \mathcal{ALCI}_{RCC3}$: There is a special role $EQ$

- Semantics:
  - “Weak”: $\text{Id}(\Delta^I) \subseteq EQ^I \Rightarrow \text{"Equality"}$
    (“EQ” is congruence relation for roles)
  - “Strong”: $\text{Id}(\Delta^I) = EQ^I \Rightarrow \text{"Identity"}$
    (as in Relation Algebras: “EQ” is congruence relation for roles and concepts)

- Further constraints, according to the RCC table
  - Reflexiveness, e.g. “Overlap”
  - Symmetry, e.g. “Externally Connected”
  - Anti-symmetry and irreflexiveness, e.g. “Proper Part”
\( \mathcal{ALCI}_{\text{RCC}3} \) is Decidable

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>( DR(a, b) )</th>
<th>( ONE(a, b) )</th>
<th>( EQ(a, b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( DR(b, c) )</td>
<td>( * )</td>
<td>{ ( DR ), ( ONE ) }</td>
<td>( DR )</td>
</tr>
<tr>
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<td>( * )</td>
<td>( ONE )</td>
</tr>
<tr>
<td>( EQ(b, c) )</td>
<td>( DR )</td>
<td>( ONE )</td>
<td>( EQ )</td>
</tr>
</tbody>
</table>

With the strong \( EQ \) semantics, an easy translation into \( \mathcal{F}_2(=) \) can be given: simply replace “EQ” in \( C \) with “\( = \)”

\[
\phi_x(C_{EQ \leftarrow \rightarrow}) \land \forall x, y : DR(x, y) \oplus ONE(x, y) \oplus x = y \land \\
\forall x, y : DR(x, y) \iff DR(y, x) \land \\
\forall x, y : ONE(x, y) \iff ONE(y, x)
\]
**ALCIT_{RCC3} is Decidable (2)**

- With the weak \(EQ\)-semantics, things are not so obvious
- Not every complete, \(\{DR, ONE, EQ\}\)-edge-colored graph is a model for the role box axioms
- We have to verify that

\[
\forall x, y, z : EQ(x, z) \Leftrightarrow DR(x, y) \land DR(y, z) \oplus OCT(x, y) \land OCT(y, z) \oplus EQ(x, y) \land EQ(y, z)
\]

holds, using only two variables
- Idea: use “\(=\)” to enforce network consistency, but take care of the fact that “\(=\)”-connected objects may have different propositional descriptions
\(\mathcal{ALCI}_{RCC^3}\) is Decidable (3)

- Nodes in \(EQ\)-clique have equivalent modal point of view
- May have different propositional descriptions
- Left structure needs three, right structure only two variables for description
- No finite model property
- $\textit{ALCI}_{\textit{RCC}5}$: $PP$, $PPI$
- $\textit{ALCI}_{\textit{RCC}8}$: $TPP$, $TPPI$, $NTPP$, $NTPPI$
- $\textit{ALCI}_{\textit{RCC}8}$ somehow allows the distinction of a role and its transitive orbit ($\rightarrow$ “PDL binary counter” concept possible)
- This seems to be impossible in $\textit{ALCI}_{\textit{RCC}5}$
The Concept \textit{even\_odd\_chain}  

\begin{align*}
\text{even\_odd\_chain} &= \text{def} \\
&\quad \text{even} \sqcap \\
&\quad (\exists \text{TPPI.}\exists \text{TPPI.} \top) \sqcap \\
&\quad (\text{even} \Rightarrow \forall \text{TPPI.} \text{odd}) \sqcap \\
&\quad (\text{odd} \Rightarrow \forall \text{TPPI.} \text{even}) \sqcap \\
&\quad (\forall \text{NTPPI.}( (\text{even} \Rightarrow \forall \text{TPPI.} \text{odd}) \sqcap \\
&\quad \quad \quad (\text{odd} \Rightarrow \forall \text{TPPI.} \text{even})))) \sqcap \\
&\quad (\forall \text{TPPI.}( (\text{even} \Rightarrow \forall \text{TPPI.} \text{odd}) \sqcap \\
&\quad \quad \quad (\text{odd} \Rightarrow \forall \text{TPPI.} \text{even})))) \sqcap \\
&\quad (\forall \text{NTPPI.}\exists \text{TPPI.} \top) \\
&\quad (\text{TPPI}^\uparrow \uparrow - \text{TPPI}^\uparrow) \subseteq \text{NTPPI}^\uparrow
\end{align*}
Is it Possible to Represent Grids?

Michael Wessel, April 2002
Is it Possible to Represent Grids? (2)

Michael Wessel, April 2002
Even though infinite grid-like models exist, we found no way to enforce the coincidence of the $x \circ y$- and $y \circ x$-successors.
Finite Model Reasoning with $\mathcal{ALC}T_{RCC5}$?  

- $\mathcal{ALC}T_{RCC5}$ contains the “proper part” role $PP$
- Question:
  Suppose we disallow the use of $PP$ in concepts – then, do we have the finite model property back?
- Answer: No! Counter example:

\[
\exists DR.T \sqcap \\
\forall DR. (\exists PO.\exists DR.C \sqcap \\
\hspace{1cm} \forall PO. \neg C \sqcap \\
\hspace{1cm} \forall DR. \neg C)
\]

$\Rightarrow$ There does not seem to be a way to tell, syntactically, whether a concept admits a finite model

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Future Work

- Check out results from “Algebraic Logic”
  - Representability of Relation Algebras (RAs) is, generally, undecidable
    * There can not be a (decidable) $\mathcal{ALCI}_{\mathcal{RA}}$ with arbitrary role boxes
  - So is the equational theory of arbitrary RAs
  - Decidable classes of (relation) algebras that are useful for spatial reasoning with DLs?

- Multi-dimensional modal logics

- Arrow-logic