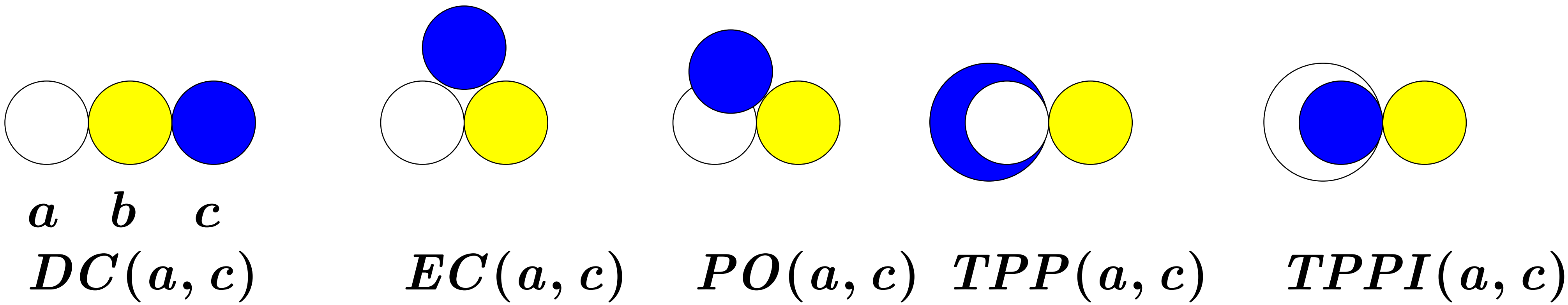
On Spatial Reasoning with Description Logics

- Motivation
- The family of $\mathcal{ALCI}_{\mathcal{RCC}}$ logics
- Work in progress
  - What we know
  - What we don't know
- Future Work

- We want a DL for "qualitative composition-table based spatial reasoning" in the style of $\mathcal{ALCRP}(\mathcal{S}_2)$, but without syntax-restrictions (if possible)
- With roles corresponding to RCC relationships
- Cohn '93: Multi-modal spatial logic with "$\Box_R, \Diamond_R$" for each RCC-relationship R
- Purely relational semantics (no truly spatial interpretations yet)
- Related to Relation Algebras, but weaker semantics (e.g., our models must not necessarily be representations of finite relation algebras)

The $\mathcal{ALCI}_{\mathcal{RCC}}$-family

- We are considering this problem in a DL-setting
- In contrast to previous work: inverse roles
- $\mathcal{ALCI}$ with disjoint roles and global role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$
- Semantics:
  $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ iff
  $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$
- With role boxes corresponding to RCC1, RCC2, RCC3, RCC5, RCC8: "$\mathcal{ALCI}_{\mathcal{RCC}}$-family", $\mathcal{ALCI}_{\mathcal{RCC}1}$, $\mathcal{ALCI}_{\mathcal{RCC}2}$, ..., $\mathcal{ALCI}_{\mathcal{RCC}8}$
- With arbitrary role boxes: undecidable (representability of Relation Algebras is undecidable)



Given $EC(a,b)$, $EC(b,c)$, what do we know about the relationship between a and c? Lookup $EC \circ EC$ in the RCC8 composition-table:

$$\forall x, y, z : EC(x,y) \wedge EC(y,z) \Rightarrow (DC(x,z) \vee EC(x,z) \vee PO(x,z) \vee TPP(x,z) \vee TPPI(x,z))$$

$$EC \circ EC \sqsubseteq DC \sqcup EC \sqcup PO \sqcup TPP \sqcup TPPI$$

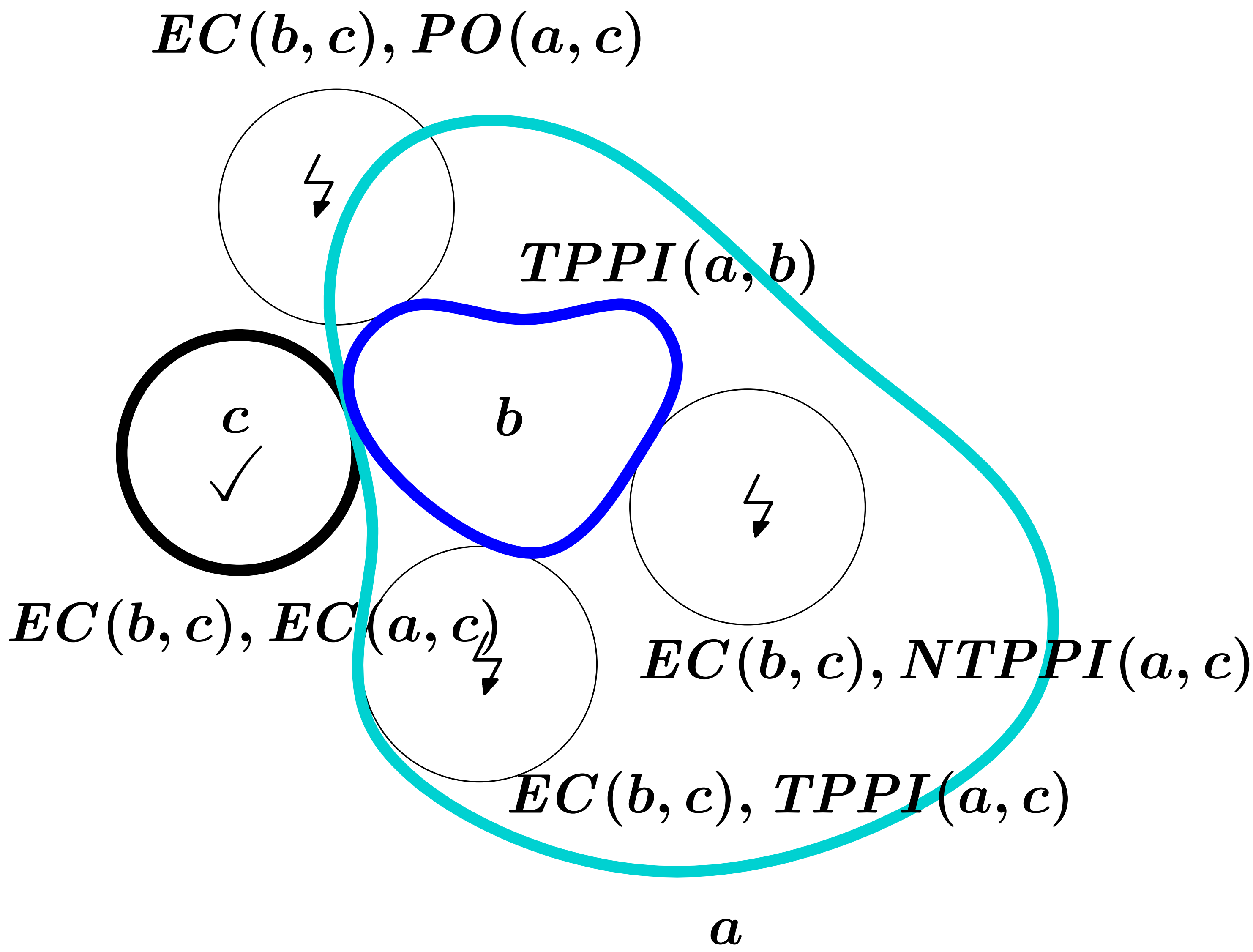
$$
\begin{array}{lcl}
circle & \dot{\sqsubseteq} & figure \\
figure\_touching\_a\_figure & \doteq & figure \sqcap \exists EC.figure \\
special\_figure & \doteq & figure \sqcap \\
 & & \forall PO.\neg figure \sqcap \\
 & & \forall NTPPI.\neg figure \sqcap \\
 & & \forall TPPI.\neg circle \sqcap \\
 & & \exists TPPI.(figure \sqcap \exists EC.circle)
\end{array}
$$

$special\_figure \sqsubseteq figure\_touching\_a\_figure$ **iff**

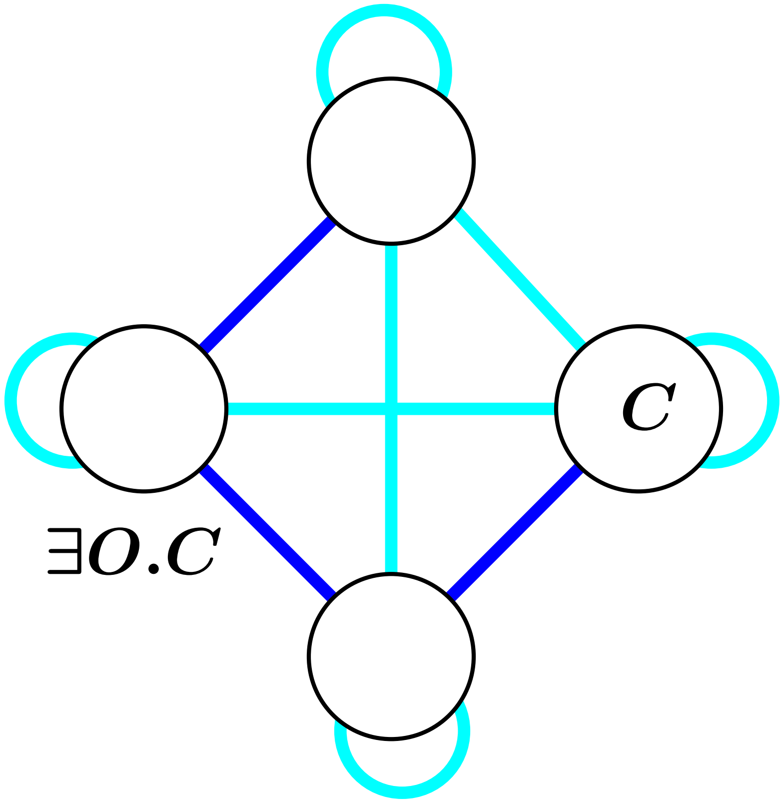
$figure \sqcap \forall PO.\neg figure \sqcap \forall NTPPI.\neg figure \sqcap \forall TPPI.\neg circle \sqcap \exists TPPI.(figure \sqcap \exists EC.circle) \sqcap \neg(figure \sqcap \exists EC.figure)$

is unsatisfiable w.r.t.

$\mathfrak{R} = \{\ldots, TPPI \circ EC \sqsubseteq EC \sqcup PO \sqcup TPPI \sqcup NTPPI, \ldots\}$


$EC(b,c), PO(a,c)$
$TPPI(a,b)$
c
b
$EC(b,c), EC(a,c)$
$EC(b,c), NTPPI(a,c)$
$EC(b,c), TPPI(a,c)$
a

- **"RCC1": Only one spatial role SR, "spatially related"**
- **Composition table: $\{SR \circ SR \rightarrow SR\}$**
- **SR is an equivalence relation**
- **Equivalent to modal logic "S5"**
- **"S5" reduction principles:**
  $\Diamond p \equiv \Box\Diamond p$, $\Box p \equiv \Diamond\Box p$, $\Diamond p \equiv \Diamond\Diamond p$, $\Box p \equiv \Box\Box p$
  $\Rightarrow$ nested occurrences of modalities can be flattened
- **NP-complete satisfiability problem**



- "RCC2": reflexive, symmetric role O = "overlap", irreflexive and symmetric role DR = "discrete from"
- Models are fairly trivial: each complete random graph with $Id(\Delta^{\mathcal{I}}) \subseteq O^{\mathcal{I}}$ is a model of the role box
- Instead of reduction principles, we have axioms like $\exists O.C \Rightarrow \forall O.(C \sqcup \exists\{O, DR\}.C) \sqcap \forall DR.\exists\{O, DR\}.C)$
- Complexity?

- $\geq \mathcal{ALCI}_{\mathcal{RCC}3}$: There is a special role EQ
- Semantics:
  - "Weak": $Id(\Delta^{\mathcal{I}}) \subseteq EQ^{\mathcal{I}}$ $\Rightarrow$ "Equality"
    ("EQ" is congruence relation for roles)
  - "Strong": $Id(\Delta^{\mathcal{I}}) = EQ^{\mathcal{I}}$ $\Rightarrow$ "Identity"
    (as in Relation Algebras: "EQ" is congruence relation for roles and concepts)
- Further constraints, according to the RCC table
  - Reflexiveness, e.g. "Overlap"
  - Symmetry, e.g. "Externally Connected"
  - Anti-symmetry and irreflexiveness, e.g. "Proper Part"

$\circ$	$DR(a,b)$	$ONE(a,b)$	$EQ(a,b)$
$DR(b,c)$	$*$	$\{DR, ONE\}$	DR
$ONE(b,c)$	$\{DR, ONE\}$	$*$	ONE
$EQ(b,c)$	DR	ONE	EQ

With the strong EQ semantics, an easy translation into $\mathcal{F}_2(=)$ can be given: simply replace "EQ" in C with "="

$$\phi_x(C_{EQ\leftarrow=}) \wedge \begin{array}{l} \forall x,y : DR(x,y) \oplus ONE(x,y) \oplus x = y \wedge \\ \forall x,y : DR(x,y) \Leftrightarrow DR(y,x) \wedge \\ \forall x,y : ONE(x,y) \Leftrightarrow ONE(y,x) \end{array}$$

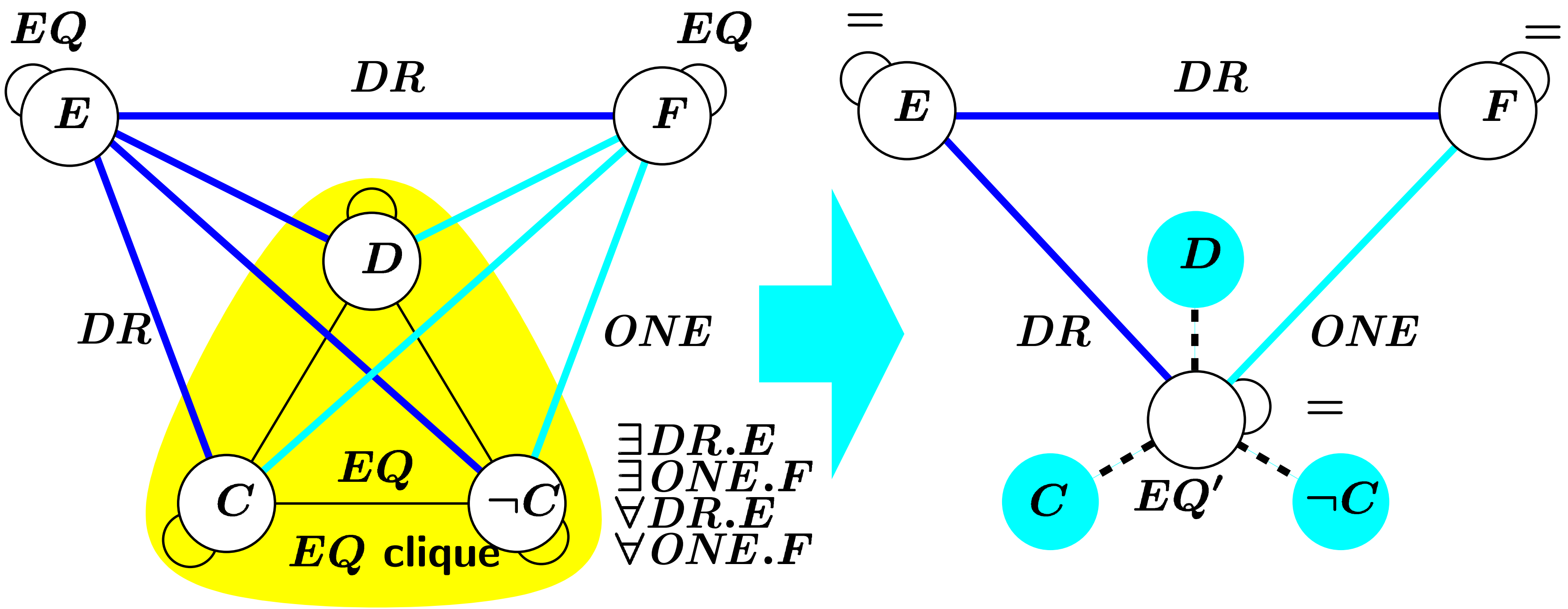
$\mathcal{ALCI}_{\mathcal{RCC}3}$ is Decidable (2)

- **With the weak EQ-semantics, things are not so obvious**
- **Not every complete, $\{DR, ONE, EQ\}$-edge-colored graph is a model for the role box axioms**
- **We have to verify that**

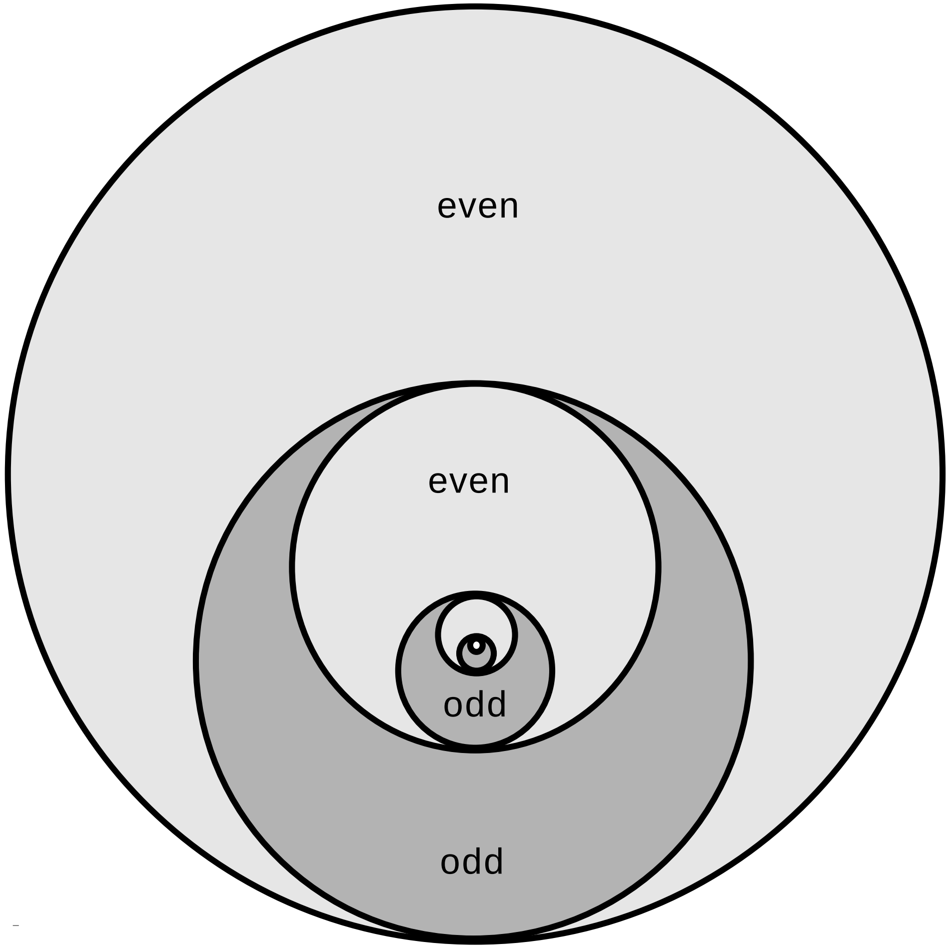
$$\begin{aligned}\forall x, y, z : EQ(x, z) \Leftrightarrow \; & DR(x, y) \wedge DR(y, z) \oplus \\ & ONE(x, y) \wedge ONE(y, z) \oplus \\ & EQ(x, y) \wedge EQ(y, z)\end{aligned}$$

holds, using only two variables

- **Idea: use "$=$" to enforce network consistency, but take care of the fact that "$=$"-connected objects may have different propositional descriptions**



- Nodes in EQ-clique have equivalent modal point of view
- May have different propositional descriptions
- Left structure needs three, right structure only two variables for description

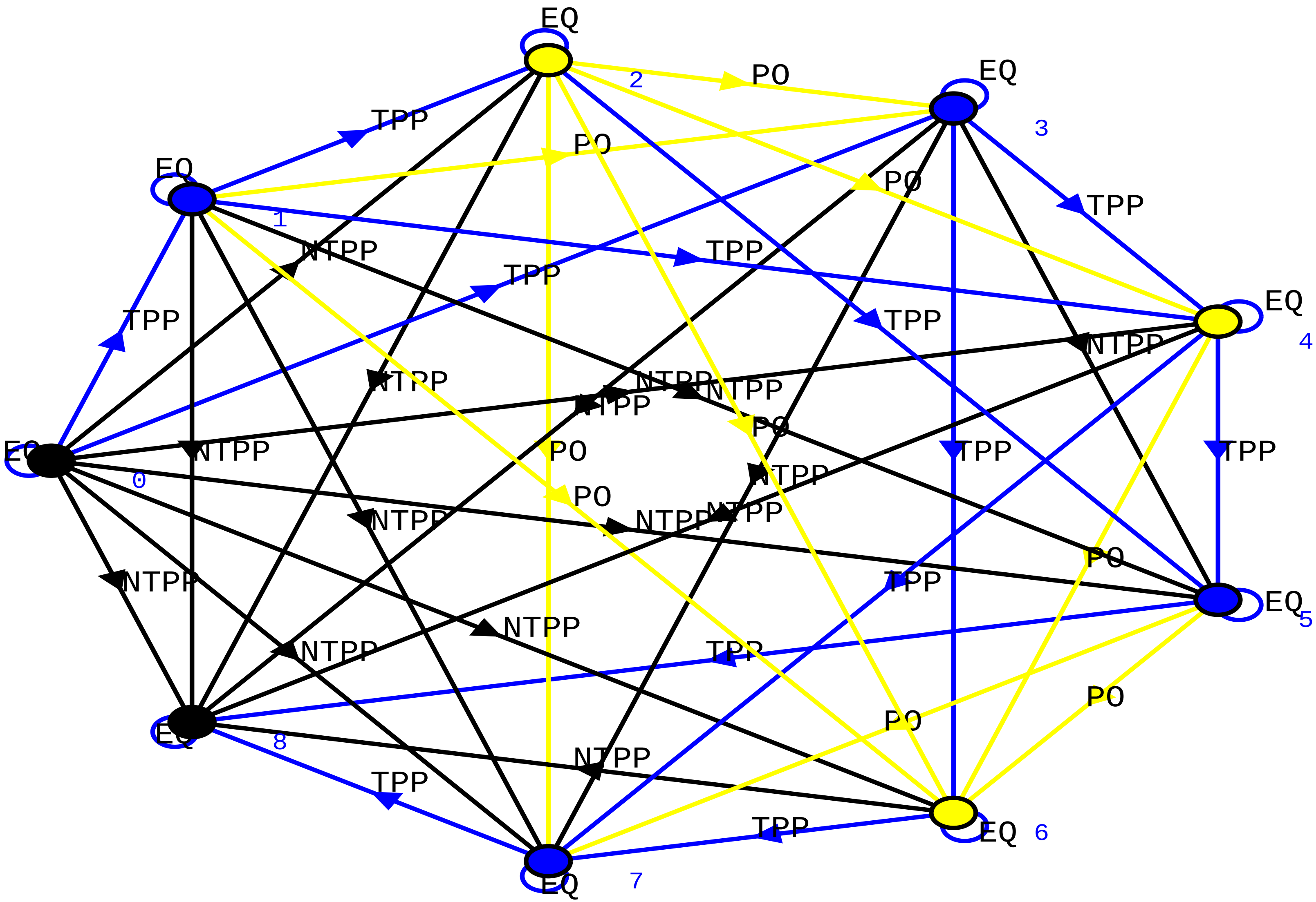


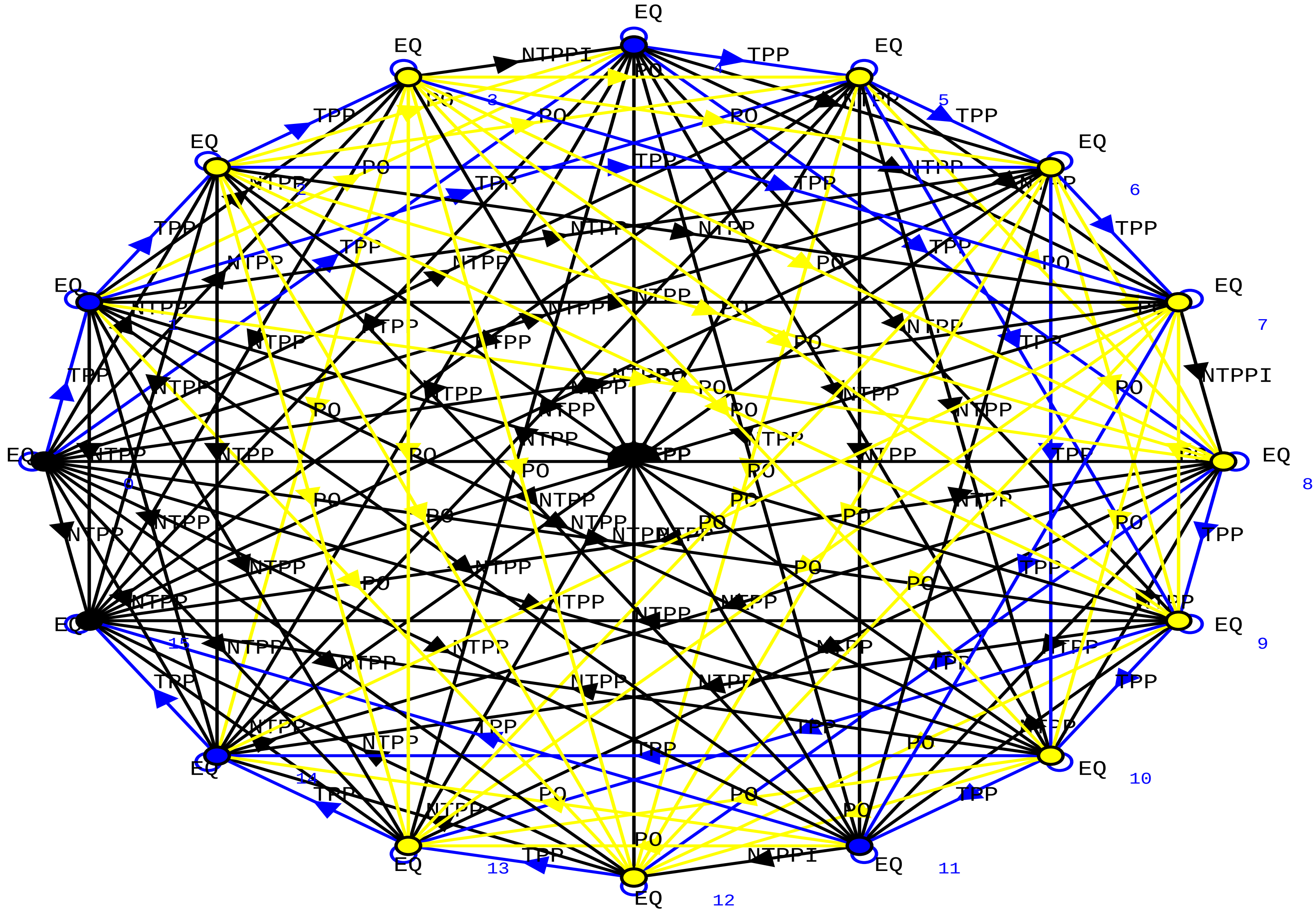
- **No finite model property**
- $\mathcal{ALCI}_{\mathcal{RCC}5}$: PP, PPI
- $\mathcal{ALCI}_{\mathcal{RCC}8}$: $TPP, TPPI, NTPP, NTPPI$
- $\mathcal{ALCI}_{\mathcal{RCC}8}$ **somehow allows the distinction of a role and its transitive orbit (→ "PDL binary counter" concept possible)**
- **This seems to be impossible in** $\mathcal{ALCI}_{\mathcal{RCC}5}$

$$
\begin{aligned}
&\textit{even\_odd\_chain} =_{def} \\
&\quad \textit{even} \sqcap \\
&\quad (\exists TPPI.\exists TPPI.\top) \sqcap \\
&\quad (\textit{even} \Rightarrow \forall TPPI.\textit{odd}) \sqcap \\
&\quad (\textit{odd} \Rightarrow \forall TPPI.\textit{even}) \sqcap \\
&\quad (\forall NTPPI.(\ (\textit{even} \Rightarrow \forall TPPI.\textit{odd}) \sqcap \\
&\quad\quad (\textit{odd} \Rightarrow \forall TPPI.\textit{even}))) \sqcap \\
&\quad (\forall TPPI.(\ (\textit{even} \Rightarrow \forall TPPI.\textit{odd}) \sqcap \\
&\quad\quad (\textit{odd} \Rightarrow \forall TPPI.\textit{even}))) \sqcap \\
&\quad (\forall NTPPI.\exists TPPI.\top)
\end{aligned}
$$

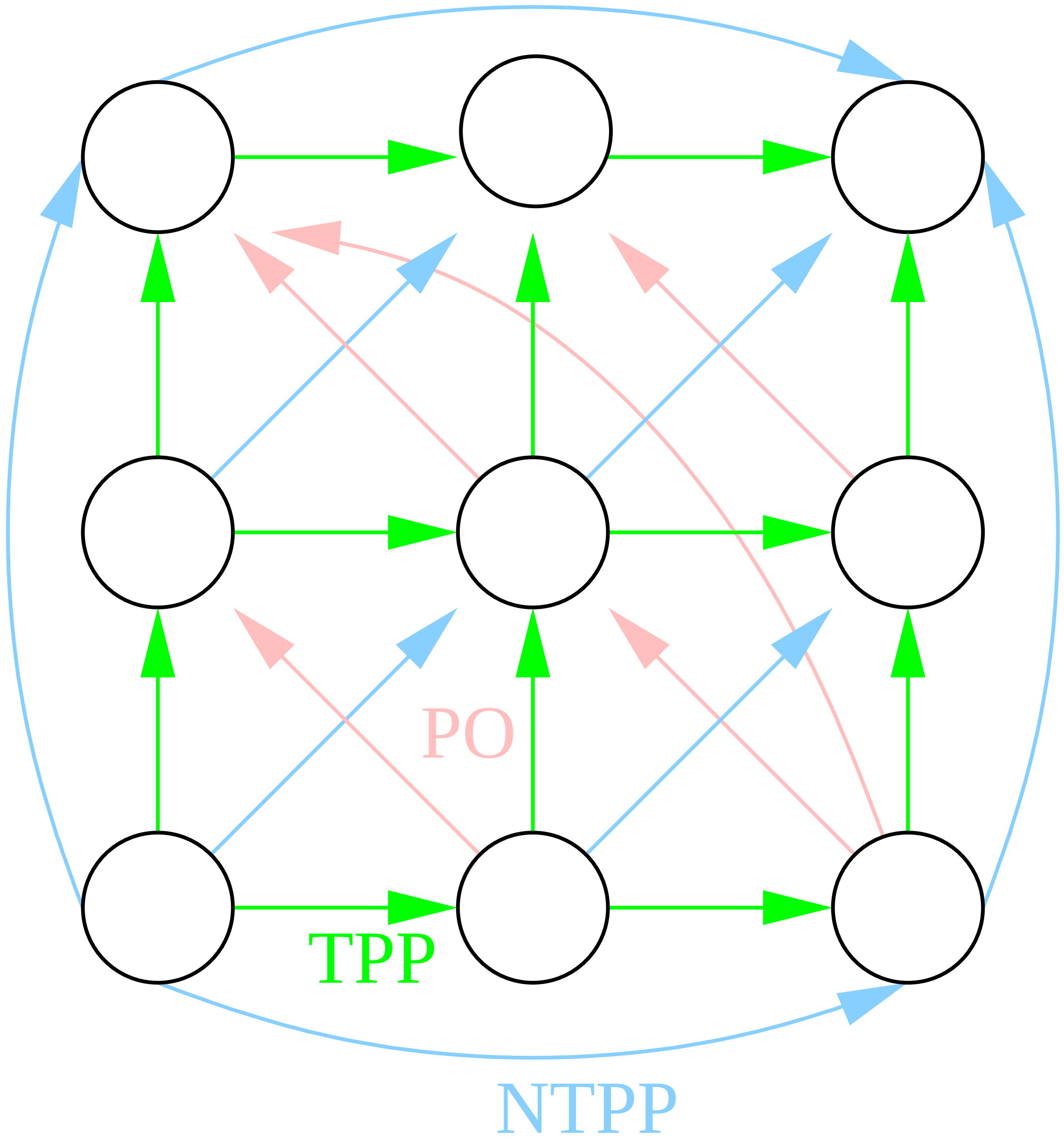
$$((TPPI^{\mathcal{I}})^{+} - TPPI^{\mathcal{I}}) \subseteq NTPPI^{\mathcal{I}}$$

Is it Possible to Represent Grids?





EQ
NTPPI
TPP
NTPP
PO
3
5
6
7
8
9
10
11
12
13
14
15



Even though infinite grid-like models exists, we found no way to enforce the coincidence of the

$x \circ y$- and $y \circ x$-

successors.

- $\mathcal{ALCI}_{\mathcal{RCC}5}$ **contains the "proper part" role** PP
- **Question:**
  Suppose we disallow the use of PP **in concepts – then, do we have the finite model property back?**
- **Answer: No! Counter example:**

$$
\begin{aligned}
&\exists DR.\top \sqcap \\
&\forall DR.(\ \exists PO.\exists DR.C \sqcap \\
&\qquad\quad \forall PO.\neg C \sqcap \\
&\qquad\quad \forall DR.\neg C)
\end{aligned}
$$

$\Rightarrow$ **There does not seem to be a way to tell, syntactically, whether a concept admits a finite model**

- Check out results from “Algebraic Logic”
  - Representability of Relation Algebras (RAs) is, generally, undecidable
    * There can not be a (decidable) $\mathcal{ALCI}_{\mathcal{RA}}$ with arbitrary role boxes
  - So is the equational theory of arbitrary RAs
  - Decidable classes of (relation) algebras that are useful for spatial reasoning with DLs?
- Multi-dimensional modal logics
- Arrow-logic