A Stick-Breaking Reduction Framework for the Goldbach Conjecture

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Abstract

We present a formal reduction framework that recasts the Goldbach conjecture as a one-dimensional stick-breaking problem. We establish two reductions: one connecting primality testing to the irreducibility of logarithmic sticks, and one connecting the Goldbach conjecture to the bisection of integer-length sticks into prime-length pieces. We prove the equivalence of these formulations, develop a density-based heuristic using the Prime Number Theorem, and identify the precise analytical barrier that separates the heuristic from a proof.

1 Introduction

The Goldbach conjecture (1742) asserts that every even integer $n \geq 4$ can be written as the sum of two prime numbers. Despite verification up to 4×10^{18} and strong heuristic evidence, no proof is known.

In this paper, we develop a geometric framework—*stick-breaking*—that recasts both prime factorization and the Goldbach conjecture as one-dimensional packing problems. While the reductions themselves do not resolve the conjecture, they provide a unified geometric language that clarifies the relationship between multiplicative and additive structure in the primes, and they make precise where a proof must overcome the independence barrier.

2 Preliminaries

We denote by $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$ the set of prime numbers. For $x > 0$, the prime counting function is $\pi(x) = |\{p \in \mathbb{P} : p \leq x\}|$.

Theorem 1 (Prime Number Theorem)**.** *As $x \to \infty$,*

$$\pi(x) \sim \frac{x}{\ln x}.$$

Theorem 2 (Fundamental Theorem of Arithmetic)**.** *Every integer $n \geq 2$ has a unique representation (up to ordering) as a product of primes:*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad p_1 < p_2 < \cdots < p_k \in \mathbb{P}, \quad a_i \geq 1.$$

Conjecture 3 (Goldbach)**.** *For every even integer $n \geq 4$, there exist primes $p, q \in \mathbb{P}$ such that $n = p + q$.*

3 The Stick-Breaking Framework

3.1 Sticks and Breaks

Definition 4 (Stick)**.** A *stick* of length $\ell > 0$ is the closed interval $[0, \ell] \subset \mathbb{R}$.

Definition 5 (Break and Pieces)**.** A *k-break* of a stick $[0, \ell]$ is a sequence of cut points $0 = c_0 < c_1 < c_2 < \cdots < c_k = \ell$, producing k *pieces* of lengths $\ell_i = c_i - c_{i-1}$ for $i = 1, \ldots, k$, satisfying $\sum_{i=1}^{k} \ell_i = \ell$.

Definition 6 (Prime-Valid Break)**.** A break is *prime-valid* if every piece length ℓ_i belongs to a specified set of admissible lengths $\mathcal{A} \subset \mathbb{R}_{>0}$.

3.2 Reduction I: Primality via Logarithmic Stick-Breaking

Definition 7 (Logarithmic Prime Lengths)**.** Define the set of *logarithmic prime lengths*:

$$\mathcal{L} = \{\ln p : p \in \mathbb{P}\}.$$

Proposition 8 (Primality Reduction)**.** *Let $n \geq 2$ be a positive integer. Then n is composite if and only if the stick $[0, \ln n]$ admits a k-break with $k \geq 2$ into pieces whose lengths all belong to $\mathcal{L}$.*

Equivalently, n is prime if and only if the stick $[0, \ln n]$ admits no break into two or more $\mathcal{L}$-pieces.

Proof. ($\Rightarrow$) Suppose n is composite. By Theorem 2, $n = p_1^{a_1} \cdots p_k^{a_k}$ with $\sum a_i \geq 2$. Taking logarithms:

$$\ln n = a_1 \ln p_1 + a_2 \ln p_2 + \cdots + a_k \ln p_k.$$

This gives a break of $[0, \ln n]$ into $\sum a_i \geq 2$ pieces, each of length $\ln p_j$ for some $p_j \in \mathbb{P}$, hence all in $\mathcal{L}$.

($\Leftarrow$) Suppose the stick $[0, \ln n]$ admits a break into pieces $\ln q_1, \ln q_2, \ldots, \ln q_m$ with $m \geq 2$ and each $q_j \in \mathbb{P}$. Then

$$\ln n = \ln q_1 + \cdots + \ln q_m = \ln(q_1 q_2 \cdots q_m),$$

so $n = q_1 q_2 \cdots q_m$. Since $m \geq 2$ and each $q_j \geq 2$, we have n composite.

If n is prime, its only factorization is $n = n$ itself, giving $\ln n = \ln n$ as a single piece—i.e., no break (the stick remains whole). □

Remark 9. The logarithm converts the multiplicative structure of $\mathbb{Z}$ into additive structure on $\mathbb{R}_{>0}$. This is the sense in which Reduction I is natural: the Fundamental Theorem of Arithmetic becomes a statement about additive decomposition of real numbers into elements of $\mathcal{L}$.

3.3 Reduction II: Goldbach via Integer Stick-Breaking

Definition 10 (Integer Prime Lengths)**.** Define the set of *integer prime lengths*:

$$\mathcal{P} = \{p \in \mathbb{N} : p \in \mathbb{P}\} \subset \mathbb{R}_{>0}.$$

Proposition 11 (Goldbach Reduction)**.** *Let $n \geq 4$ be an even integer. Then the following are equivalent:*

(i) n is expressible as the sum of two primes: $n = p + q$, $p, q \in \mathbb{P}$.

(ii) The stick $[0, n]$ admits a 2-break into pieces whose lengths both belong to $\mathcal{P}$.

Consequently, the Goldbach conjecture (Conjecture 3) is equivalent to the assertion that every stick of even integer length $n \geq 4$ admits a 2-break into $\mathcal{P}$-pieces.

Proof. A 2-break of $[0, n]$ produces pieces of lengths p and $n - p$ with $0 < p < n$. Both pieces belong to $\mathcal{P}$ if and only if $p \in \mathbb{P}$ and $n - p \in \mathbb{P}$, i.e., $n = p + (n - p)$ is a sum of two primes. The equivalence is immediate. □

3.4 Comparison of the Two Reductions

	Reduction I (Primality)	**Reduction II** (Goldbach)
Stick length	$\ln n$	n
Admissible piece lengths	$\mathcal{L} = \{\ln p : p \in \mathbb{P}\}$	$\mathcal{P} = \mathbb{P}$
Number of pieces	≥ 2 (composite) or 1 (prime)	exactly 2
Arithmetic operation	multiplication ($n = \prod p_i$)	addition ($n = p + q$)
Guaranteed solvability	Yes (Thm. 2)	Open (Conj. 3)

Remark 12 (The Additive–Multiplicative Duality)**.** Reduction I succeeds unconditionally because the Fundamental Theorem of Arithmetic guarantees that every integer has a multiplicative decomposition into primes. The logarithm converts this multiplicative certainty into an additive decomposition of a real-valued stick.

Reduction II asks for an additive decomposition directly in the integers. No analogue of the Fundamental Theorem guarantees that every even integer has such a decomposition. The central tension of the Goldbach conjecture is precisely the gap between the certainty of multiplicative structure and the apparent—but unproven—ubiquity of additive structure among the primes.

4 Density Analysis: Why the Stick Should Always Break

We now make precise the heuristic argument that Goldbach pairs are plentiful.

Definition 13 (Goldbach Counting Function)**.** For even $n \geq 4$, define

$$G(n) = |\{p \in \mathbb{P} : p \leq n/2 \text{ and } n - p \in \mathbb{P}\}|.$$

The Goldbach conjecture is equivalent to $G(n) \geq 1$ for all even $n \geq 4$.

4.1 The Independence Heuristic

Proposition 14 (Hardy–Littlewood Heuristic)**.** *If we model the events "p is prime" and "$n-p$ is prime" as independent, each occurring with probability $\sim 1/\ln(n/2)$, then*

$$G(n) \approx \sum_{\substack{3\le p\le n/2\\ p \text{ odd}}} \frac{1}{\ln p}\cdot\frac{1}{\ln(n-p)}.$$

Approximating $\ln p \approx \ln(n-p) \approx \ln(n/2)$ and summing over $\sim n/4$ odd candidates:

$$G(n) \approx \frac{n/4}{(\ln(n/2))^2} \sim \frac{n}{4(\ln n)^2}.$$

The refined Hardy–Littlewood conjecture incorporates a correction for local divisibility constraints:

Conjecture 15 (Hardy–Littlewood, 1923)**.** *For even $n\to\infty$,*

$$G(n) \sim 2C_2 \prod_{\substack{p\mid n\\ p>2}} \frac{p-1}{p-2}\cdot\frac{n}{(\ln n)^2},$$

where $C_2 = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \approx 0.6601\ldots$ is the twin prime constant.

4.2 Growth of Valid Cuts

Proposition 16 (Unbounded Growth of $G(n)$, Heuristic)**.** *Under the Hardy–Littlewood heuristic, $G(n)\to\infty$ as $n\to\infty$. More precisely, for all even $n\ge 4$:*

$$G(n) \ge C\cdot\frac{n}{(\ln n)^2}$$

for some absolute constant $C>0$.

Remark 17 (Stick-Breaking Interpretation)**.** In the stick-breaking language: as the stick grows longer, the number of valid cut positions grows like $n/(\ln n)^2$. For $n=10^6$, this predicts approximately 1,300 valid cuts. The stick becomes *easier* to break as it grows, not harder. Any proof of Goldbach must formalize this observation.

5 The Barrier: From Heuristic to Proof

5.1 What the Heuristic Assumes

The density argument of Section 4 treats the primality of p and $n-p$ as independent events. This assumption fails because:

(a) **Parity constraint.** For n even and p odd prime, $n-p$ is automatically odd—a prerequisite for primality (except $p=2$). This is a *helpful* correlation already accounted for.

(b) **Small prime divisibility.** If $3 \mid n$, then $p \equiv 0 \pmod 3$ forces $3 \mid (n-p)$, so $n-p$ cannot be prime (unless $n-p=3$). The correction factor $\prod_{p \mid n}(p-1)/(p-2)$ in Conjecture 15 compensates for this.

(c) **Higher-order correlations.** Primes exhibit subtle correlations in arithmetic progressions (Bombieri–Vinogradov theorem, Elliott–Halberstam conjecture) that are difficult to control uniformly.

5.2 Known Rigorous Results

The strongest results toward Goldbach are:

Theorem 18 (Chen, 1966/1973)**.** *Every sufficiently large even integer can be written as $p+m$, where p is prime and m is either prime or a product of two primes (a semiprime).*

This can be interpreted in stick-breaking terms: the stick can always be broken in two, with one piece of prime length and the other of *almost-prime* length (≤ 2 prime factors).

Theorem 19 (Vinogradov, 1937; Helfgott, 2013)**.** *Every odd integer $n \geq 7$ is the sum of three primes.*

In stick-breaking terms: every stick of odd length ≥ 7 can be broken into three prime-length pieces. The "extra piece" provides enough flexibility to close the minor arc estimates.

5.3 The Circle Method and Stick-Breaking

The analytical tool closest to the stick-breaking framework is the Hardy–Littlewood circle method. Define the prime exponential sum:

$$f(\alpha) = \sum_{\substack{p \leq n \\ p \in \mathbb{P}}} e^{2\pi i p \alpha}.$$

Then the number of Goldbach representations is:

$$G(n) = \int_0^1 f(\alpha)^2 \, e^{-2\pi i n \alpha} \, d\alpha.$$

Remark 20 (Fourier Interpretation of Stick-Breaking)**.** The integral above "scans" all possible cut positions simultaneously. The function $f(\alpha)^2$ encodes all pairs of prime-length pieces, and the factor $e^{-2\pi i n \alpha}$ selects those pairs summing to n.

- **Major arcs** (where $\alpha \approx a/q$ with small q): Contribute the main term $\sim C \cdot n/(\ln n)^2$, confirming the density heuristic.
- **Minor arcs** (all other α): Must be shown to contribute $o(n/(\ln n)^2)$, i.e., they cannot cancel the main term.

For three primes ($f(\alpha)^3$), Vinogradov proved minor arc cancellation using the extra averaging from the third factor. For two primes, the minor arcs remain uncontrolled. *This is the precise barrier.*

6 Toward a Proof: Open Directions

The stick-breaking framework suggests several avenues:

6.1 Direction 1: Measure-Theoretic Stick-Breaking

Consider the set of "prime marks" on $[0, n]$:

$$S_n = \{p \in \mathbb{P} : p \leq n\}.$$

Define the "Goldbach set" $\mathcal{G}_n = S_n \cap (n - S_n)$, where $n - S_n = \{n - p : p \in S_n\}$. Goldbach holds for n iff $\mathcal{G}_n \neq \emptyset$.

By PNT, $|S_n| \sim n/\ln n$. Heuristically, the "density" of S_n in $[1, n]$ is $\sim 1/\ln n$, and similarly for $n - S_n$. If these sets were "generic" (in an appropriate measure-theoretic sense), their intersection would be non-empty whenever their combined density exceeds a threshold. Formalizing a notion of "generic distribution" for primes—one that captures Bombieri–Vinogradov equidistribution—could yield a proof.

6.2 Direction 2: Additive Combinatorics

By the Cauchy–Davenport theorem, for $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$:

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

The primes modulo small numbers are well-distributed (Dirichlet's theorem). An approach via sumset estimates in $\mathbb{Z}/n\mathbb{Z}$, combined with sieve bounds on prime density, might establish that $\mathbb{P} + \mathbb{P}$ covers all even integers.

6.3 Direction 3: Tiling and Quasicrystalline Structure

The Pisot–Vijayaraghavan connection noted in the original discussion relates to the fact that certain algebraic integers generate self-similar tilings whose tile lengths encode multiplicative number-theoretic information. If a tiling framework could simultaneously encode both the additive and multiplicative structure of primes—bridging Reductions I and II—it might provide new constraints on $\mathcal{G}_n$.

7 Conclusion

We have established two formal reductions:

1. **Primality $\Leftrightarrow$ logarithmic stick irreducibility** (Proposition 8), which is unconditionally true by the Fundamental Theorem of Arithmetic.

2. **Goldbach $\Leftrightarrow$ integer stick bisection into prime pieces** (Proposition 11), which is a faithful reformulation of the conjecture.

The stick-breaking framework unifies these two problems and highlights the additive–multiplicative duality (Remark 12). The density analysis shows that the number of valid cuts grows as $n/(\ln n)^2$, strongly suggesting that Goldbach holds. The precise barrier to a proof is controlling the minor arcs in the circle method—equivalently, showing that

the "arithmetic correlations" among primes cannot conspire to eliminate all valid cut positions.

The framework does not resolve the conjecture, but it provides a clean geometric language in which the problem, the evidence, and the barrier are all visible.

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